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The qualitative analysis of composite systems

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TABLE OF CONTENTS

CHAPTER ONE: INTRODUCTION

Review of Previous Work

One of the most important topics in the qualitative analysis of physical systems, and the one which chiefly concerns us here, is that of stability. Two basic notions of stability have been widely studied in systems theory: Lyapunov stability and input-output stability. The former traces its origin back to the work of Lyapunov in the late nineteenth century -- although its applications to control systems only became known in the United States much more recently[1,2]. In order to discuss Lyapunov stability, one must first obtain a time-domain representation of the system under study in terms of state variables. Lyapunov stability concerns the behavior of the undriven system (i.e., the system with all inputs set equal to zero) in response to changes in the initial conditions on the state variables about a set of equilibrium values (conventionally taken to be the origin in state space). Roughly speaking, if for sufficiently small, but otherwise arbitrary, variations in these initial conditions, the system relaxes toward, or stays near to, the equilibrium state; then the system is said to be stable in the sense of Lyapunov relative to that equilibrium.

The second notion of stability is much more recent, the pioneering work of Zames[3-7] and Sandberg[8-14] having appeared only a decade ago. (Two recent summaries of this early work and subsequent progress are the books of Willems[15] and Holtzman[16].) In the study of input-output stability, a system is regarded as a relation between one function space (of inputs) and a second function space (of outputs). The system is said to be input-output stable if this relation is bounded and continuous. Roughly speaking, the requirement of boundedness implies that inputs of finite "size" result in outputs of finite "size." (More precisely, boundedness means that the ratio of output size to input size does not exceed a fixed, finite value -- i.e., that the system has finite gain.) The notion of "size" is given precise meaning as a norm on a function space. Similarly, the requirement of continuity implies that small changes in system inputs result in correspondingly small changes in outputs. Thus, the outputs of a bounded system can not grow without bound if the inputs do not, while the outputs of a continuous system can not be critically sensitive to changes in the inputs (e.g., to input noise).

The concept of input-output stability has a certain intuitive appeal to the engineer since it is essentially a formal statement (albeit, in a rather abstract context) of the attributes usually conveyed by the words "well-behaved" or "stable." In contrast, the physical implications of the Lyapunov concept (of stability with respect to changes in initial conditions) -- although familiar from the study of differential equations, are not as clear in many practical engineering situations since most systems of interest are driven. On the other hand, input-output stability has an inherent global aspect which may be a limitation in practical applications: a system is either stable with respect to a certain class of inputs, or it is not; whereas, for Lyapunov stability, a single system may possess both stable and unstable equilibria, each holding sway in its own local domain of initial conditions.

The study of the stability of physical systems has dealt primarily with two problems: (1) The analysis problem -- given a particular system, is it stable or not? (2) The design problem -- given an unstable system (or a system which is stable but possesses certain undesirable properties), how can it be modified to achieve stability (or to improve its other properties while preserving stability)?

For simple linear systems, such as those governed by ordinary differential or difference equations with constant coefficients, these problems have been thoroughly investigated. For such systems, there are straightforward methods of deciding whether a particular system is stable or unstable (associated with the names Routh-Hurwitz, Jury, Nyquist, Bode, etc.) and numerous, widely used, methods of compensation to achieve stability and satisfy other design criteria (e.g., root-locus, Bode, and Nyquist compensation, s- and z-plane synthesis). Furthermore, for such simple systems, the two stability concepts discussed above are essentially equivalent.

The situation for nonlinear systems is quite different. In spite of a quarter century of effort, no generally applicable methods of stability analysis or compensation exist for nonlinear systems. In view of the large variety of possible nonlinearities and the difficulty of obtaining comprehensive conditions which can distinguish between stability and instability even for individual nonlinear systems, it seems unreasonable to expect that such general methods will ever be found. As an added complication, the relation between Lyapunov and input-output stability for nonlinear systems is neither simple nor well-understood. (One of the few papers to deal with this relationship is that of J. C. Willems[17].)

Some progress was made on the problem of the Lyapunov stability of nonlinear systems by Popov[18], Kalman[19], and

Yacubovich[20] (among others), who studied single-loop systems consisting of a time-invariant linear element (characterized by its Laplace or Fourier transform) in cascade with a memoryless nonlinearity. Subsequently, Sandberg[8-14] and Zames[3-7] achieved important results concerning the input-output stability of single-loop systems in which both forward and feedback paths are general (not necessarily linear or memoryless) relations on an extended function space. More recently, research has centered on more complicated systems containing many nonlinearities. These recent studies have taken one of two basic approaches: (1) A number of papers have dealt with systems which can be cast into the form of a single-loop feedback system having multiple-inputs and multiple-outputs[21-29]. In spite of the wide interest in this approach, it is probably fair to say that the resulting stability conditions are difficult to apply and have gained little currency. (2) A second group of papers resorts to a familiar tactic in the study of complicated (multiple-input multiple-output) systems: namely, these papers try to reduce the difficult problem of overall stability to a series of simpler problems by decomposing the overall system into a number of subsystems and a corresponding interconnecting structure[30-33]. By first studying the stability properties of each subsystem, these papers hope to simplify the problem of overall stability. Although the stability conditions obtained in the second group of papers may be more conservative than the corresponding conditions obtained in the first (in fact, little is known about the relative merits of these two sets of conditions), they have the important advantages that they may be checked in a straightforward manner and that (typically) all of the stability conditions but one involve the parameters of only one subsystem at a time.

It is largely due to the efforts of the researchers who wrote this second group of papers (together with parallel work in fields other than stability) that multiple-input multiple-output systems have been variously termed <u>composite</u>, <u>interconnected</u>, <u>large-scale</u>, or <u>multiple-loop</u> systems. The appropriateness of a particular choice of terminology depends more on ones point of view than on the nature of the system under study. Before expounding the point of view adopted in the present work, it is worthwhile to briefly summarize the types of results which have already been achieved on the stability of multiple-input multiple-output systems. Such results can be grouped into four broad categories, the goal in each case being to find sufficient conditions for stability:

(1) Studies of the Lyapunov stability of single-loop systems in which the stability condition (a generalization of the original Popov condition[18]) requires a certain matrix of Fourier transforms to be positive definite for all frequencies.[21-27]

(2) Studies of the L_p- and l_p-stability of single-loop systems for $1 \leq p \leq \infty$ (a type of input-output stability) in which the stability condition requires that one first find a suitable matrix K of constants and then check to see that a certain function of a complex frequency s (the form of which depends on the choice of K) is strictly positive in the region $\mathrm{Re}(s) \geq 0$.[28,29]

(3) Studies of Lyapunov stability for systems which may be viewed as an interconnection of stable (or unstable) subsystems. The resulting stability (or instability) conditions typically involve restrictions on the Lyapunov function of each subsystem together with one overall condition involving parameters which describe the interconnecting structure as well as those of each subsystem.[30-33]

(4) Studies of the input-output stability of systems which may be viewed as an interconnection of "isolated subsystems" by means of constant multipliers and summing junctions. For the purposes of such studies, each subsystem is characterized by one or two numbers which deal with its input-output properties (e.g., gains or conicity constants). Stability conditions developed so far simply require that all the successive principal minors of a certain test matrix of constants be positive,

the components of this test matrix being formed from the parameters which characterize each subsystem and those describing the interconnecting structure.[34]

Viewpoint of the Present Paper

This paper deals with the analysis and design of interconnected systems viewed in a "black box" sense as a collection of relations connecting inputs to outputs. Each input and output is assumed to belong to an appropriate extended function space. Throughout, we seek sufficient conditions for input-output stability, where stability is interpreted as boundedness and continuity of the relations which connect system inputs to each system output. In particular, we seek to extend the results on multiple-input multiple-output systems described in category (4) in the previous section.

Thus, we choose to decompose a multiple-input multiple-output system into a number of subsystems and a corresponding interconnecting structure. The present paper, however, adopts a viewpoint on interconnected systems different from those found in previous works: in this paper our primary concern is with systems which may be viewed as an interconnection of single-loop feedback systems. Each such single loop, regarded as an input-output relation in its own right, shall be termed an isolated subsystem of the overall interconnected system. The stability of the (large) interconnected system is then studied in terms of a margin of boundedness and, in some cases, a gain factor for each isolated subsystem, together with parameters describing the interconnecting structure. The margin of boundedness (first introduced in Reference 34) is a measure of the degree of stability of a particular subsystem. In many instances, this quantity has a graphical interpretation in the Nyquist (or modified frequency response) plane reminiscent of the familiar phase and gain margins of linear systems theory. Besides the advantages inherent in any approach which views a complicated system as an interconnection of simpler subsystems, the present treatment has some advantages which are unique to itself: (1) It formulates stability conditions in terms of quantities (viz., margins of boundedness) which can be intuitively understood and manipulated by the designer (often via graphical techniques). (2) It is well-suited to the stabilization and compensation of large-scale systems by means of local feedback. (3) It deals directly with the parameters of the original system, i.e., there is no need to transform the entire interconnected system as is the case when applying Theorem 3 of Reference 34.

Outline

This paper is divided into several Chapters. In Chapter Two, we give some basic notation and then introduce a number of definitions which are convenient in the subsequent discussions of continuous time and discrete time systems. In Chapter Three, we discuss single-loop multiple-input multiple-output systems. Here, new stability results are given for such systems and are compared with earlier results. We also introduce the concepts of margin of boundedness, margin of continuity, gain factor, and incremental gain factor which prove useful in formulating the later results. In Chapter Four, we give some intermediate results and note that these contain the results of Reference 34 as a special case. Chapter Five presents general results on L_2- and l_2-boundedness and continuity of interconnected feedback systems. Chapter Six reports similar results for L_∞- and l_∞-boundedness and continuity. In Chapter Seven, we present results which serve to generalize the Popov boundedness condition for single-loop systems to interconnected feedback systems. Chapter Eight discusses the relative merits of various stability conditions and offers analysis and design procedures for interconnected systems. In Chapter Nine, we discuss three examples which serve to illustrate how the various stability theorems are applied.

CHAPTER TWO: NOTATION AND DEFINITIONS

General Remarks

Let $\in$ denote set membership. Union is denoted by $\cup$ and intersection by $\cap$. The essential supremum and maximum of a set are denoted by ess sup and max, respectively. The symbol j is used for $\sqrt{-1}$ and s always denotes a complex number with real part σ and imaginary part ω. Let $R = (-\infty,\infty)$, $R^+ = [0,\infty)$, $I = \{\ldots,-2,-1,0,1,2,\ldots\}$, and $I^+ = \{0,1,2,\ldots\}$. Let R^m denote Euclidean m-space. If x and $y \in R^m$, then let $|x|$ denote the Euclidean norm of x, let $x \cdot y$ denote the Euclidean inner product of x and y, and let $x \leqslant y$ indicate that $x_i \leqslant y_i$ for each $i = 1,2,\ldots m$. Let $A = [a_{ij}]$ denote an n x m matrix with $i,j^{\underline{th}}$ component $a_{ij} \in R$. Let A' denote the transpose of A. Let I denote the n x n identity matrix, and let $[\text{diag } a_i]$ denote the n x n diagonal matrix with $i^{\underline{th}}$ diagonal component $a_i \in R$.

The notation $f: X \longrightarrow Y$ refers to the mapping f from the set X into the set Y. The notation $\{x|A\}$ is interpreted as the set of all x such that the condition A is satisfied. The Cartesian product of two sets is defined by $X \times Y = \{(x,y)| x \in X \text{ and } y \in Y\}$. If A is an n x n matrix, then the $i^{\underline{th}}$ successive principal minor of A is the determinant of the square matrix obtained from A by deleting all elements a_{kl} with either $k \geq i$ or $l \geq i$ (or both). There are n such successive principal minors if A is n x n.

Before going any further, it is worth noting that in this paper we discuss the question of the stability of the solutions of a set of equations comprising a system model without reference to the questions of the existence or uniqueness of those solutions. Thus, we separate the questions of stability and well-posedness of our system models. Therefore, the reader may adopt one of two attitudes with respect to the stability results derived here: (1) He may wish to supplement our stability conditions with further conditions which guarantee that a particular system model under study is well-posed as well as stable. (2) He may accept our conditions as they stand, with the understanding that these conditions suffice to guarantee stable behavior for all those (possibly nonunique) system solutions which do exist. In the remainder of this Chapter, we discuss the two types of systems with which we shall be concerned -- continuous time and discrete time systems.

Continuous Time Systems

Let x be an arbitrary function which maps R into R^m.

The value of x at time $t \in R$ is denoted by $x(t)$. Defining

$$\|x\|_{L_2} = \left[\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right]^{\frac{1}{2}}$$

and

$$\|x\|_{L_\infty} = \operatorname*{ess\,sup}_{t \in R} |x(t)| ,$$

the normed spaces $L_2(-\infty,\infty)$ and $L_\infty(-\infty,\infty)$ (hereafter denoted by L_2 and L_∞, respectively) are defined by

$$L_2 = \{ x : R \longrightarrow R^m \mid \|x\|_{L_2} < \infty \}$$

and

$$L_\infty = \{ x : R \longrightarrow R^m \mid \|x\|_{L_\infty} < \infty \}$$

If $x, y \in L_2$, then the inner product of x and y is defined by

$$\langle x,y \rangle_{L_2} = \int_{-\infty}^{\infty} x(t) \cdot y(t) \, dt.$$

The truncation of x at time T, denoted by x_T, is defined by

$$x_T(t) = \begin{cases} x(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}$$

Following Zames[3-5], we introduce the corresponding extended function spaces L_{2e} and $L_{\infty e}$ defined by

$$L_{2e} = \left\{ x: R \longrightarrow R^m \;\middle|\; x_T \in L_2 \text{ for all } T \in R \right\}$$

and

$$L_{\infty e} = \left\{ x: R \longrightarrow R^m \;\middle|\; x_T \in L_\infty \text{ for all } T \in R \right\}$$

For the remainder of this section, let X_e stand for either L_{2e} or $L_{\infty e}$. A multiple-input multiple-output continuous time system is modeled as a relation on the product space $X_e^n = X_e \times X_e \times \ldots \times X_e$. (The number of inputs and outputs is taken to be equal and is denoted by n.) Thus, inputs and outputs are assumed to belong to X_e. (This assumption precludes the possibility of finite escape times in the systems under consideration.) System elements are represented by relations on X_e. For such a system, stability is interpreted as boundedness and continuity of the relations which connect system inputs to each system output.

Let H be a relation on X_e with domain Do(H) and range Ra(H). A particular image of $x \in$ Do(H) under the relation provided by H is denoted by $Hx \in$ Ra(H). Such a relation is <u>causal</u> if $(Hx)_T = (Hx_T)_T$ for all $T \in R$, $x \in$ Do(H), and $Hx \in$ Ra(H). H is <u>time-invariant</u> if it commutes with all time delays and <u>memoryless</u> if the value of Hx at time t depends only on the value of x at time t.

First, consider multiple-input multiple-output continuous time systems for which the underlying extended function space is the space L_{2e}. The discussion of the boundedness of such systems is simplified by introducing a number of definitions. In each of the following six definitions, H is a relation on L_{2e} and the indicated condition is understood to apply for all $T \in R$, $x \in Do(H)$, and $Hx \in Ra(H)$. The <u>L_2-gain</u> of H, $g_{L_2}(H)$, is the smallest nonnegative number M such that $\|(Hx)_T\|_{L_2} \leq M\|x_T\|_{L_2}$. H is <u>interior conic (c,r)</u> if there are real numbers $r \geq 0$ and c such that $\|(Hx)_T - cx_T\|_{L_2} \leq r\|x_T\|_{L_2}$. H is <u>exterior conic (c,r)</u> if the preceding condition holds with the inequality reversed. (Note: A relation which is interior conic (c,r) has finite gain $g_{L_2}(H)$ which can not exceed the larger of the two numbers $|c+r|$ and $|c-r|$.) H is <u>inside the sector $\{a,b\}$</u> if for some real numbers a,b we have $\langle (Hx)_T - ax_T, (Hx)_T - bx_T \rangle_{L_2} \leq 0$. H is <u>outside the sector $\{a,b\}$</u> if the preceding condition holds with the inequality reversed. (Note: No particular ordering of a and b is implied. Thus, the statements that H is inside the sector $\{a,b\}$ and that H is inside the sector $\{b,a\}$ are equivalent. If we let $c = \frac{1}{2}(b+a)$ and $r = \frac{1}{2}|b-a|$, then H is inside (outside) the sector $\{a,b\}$ if and only if it is interior (exterior) conic (c,r).) Finally, H is positive if $\langle x_T, (Hx)_T \rangle_{L_2} \geq 0$.

In order to discuss the continuity of such systems, we introduce six corresponding incremental definitions. In each case, the indicated condition is to be understood to apply for all $T \in R$; $x, y \in Do(H)$; and $Hx, Hy \in Ra(H)$. The <u>incremental L_2-gain</u> of H, $g^I_{L_2}(H)$, is the smallest nonnegative number M such that $\|(Hx - Hy)_T\|_{L_2} \leq M\|(x - y)_T\|_{L_2}$. H is <u>incrementally interior conic (c,r)</u> if there are real numbers $r \geq 0$ and c such that $\|(Hx - Hy)_T - c(x - y)_T\|_{L_2} \leq r\|(x - y)_T\|_{L_2}$. H is <u>incrementally exterior conic (c,r)</u> if the preceding condition holds with the inequality reversed. H is <u>incrementally inside the sector $\{a,b\}$</u> if for some real numbers a,b we have $\langle (Hx - Hy)_T - a(x - y)_T, (Hx - Hy)_T - b(x - y)_T \rangle_{L_2} \leq 0$. H is <u>incrementally outside the sector $\{a,b\}$</u> if the preceding condition holds with the inequality reversed. Finally, H is <u>incrementally positive</u> if $\langle (x - y)_T, (Hx - Hy)_T \rangle_{L_2} \geq 0$. Note that if H is incrementally inside the sector $\{a,b\}$, then it is necessarily inside the sector $\{a,b\}$. Similar implications hold for each pair of incremental and nonincremental definitions.

Next, let us consider multiple-input multiple-output continuous time systems in which the underlying extended space is $L_{\infty e}$ (rather than L_{2e} as discussed above). In order to discuss the boundedness of such systems, we find it convenient to introduce several definitions. Let H be a relation on L_∞. Each of the following conditions is understood to apply for all $T \in R$, $x \in Do(H)$, and $Hx \in Ra(H)$. First, the

L_∞-gain of H, $g_{L_\infty}(H)$, is defined as the smallest non-negative number M such that $\|(Hx)_T\|_{L_\infty} \leq M\|x_T\|_{L_\infty}$. Following Zames[6], we convert functions in $L_{\infty e}$ into functions in L_{2e} by supplying an appropriate exponential weight. Denote by e^{σ} the exponential function on R which takes on the value $\exp(\sigma t)$ at time $t \in R$. Note that if $x \in L_{\infty e}$, then $x_T e^{-\sigma} \in L_2$ for any $\sigma < 0$. We define two new symbols for the norm and scalar product of truncated, weighted functions (with $\sigma < 0$ understood) by

$$\|x;T,\sigma\| = \|x_T e^{-\sigma}\|_{L_2}$$

$$\langle x,y;T,\sigma\rangle = \langle x_T e^{-\sigma}, y_T e^{-\sigma}\rangle_{L_2}$$

With these preliminaries, H is <u>interior conic (c,r) with weight σ</u> if for some real constants $r \geq 0$, $\sigma < 0$ and c, we have $\|Hx - cx;T,\sigma\| \leq r\|x;T,\sigma\|$. Similarly, H is <u>exterior conic (c,r) with weight σ</u> if the preceding condition holds with the inequality reversed. H is said to be <u>positive with weight σ</u> if for some constant $\sigma < 0$ we have $\langle x,Hx;T,\sigma\rangle \geq 0$.

Similarly, in order to discuss the continuity of such systems, we introduce several incremental definitions. The <u>incremental L_∞-gain</u> of H, $g^I_{L_\infty}(H)$, is the smallest nonnegative number M such that $\|(Hx - Hy)_T\|_{L_\infty} \leq M\|(x - y)_T\|_{L_\infty}$. (This definition, as well as all other definitions in this paragraph, is understood to imply that the indicated condition applies for all $T \in R$; $x,y \in Do(H)$; and $Hx, Hy \in Ra(H)$.) H is <u>incrementally interior conic (c,r) with weight σ</u> if for some real constants $r \geq 0$, $\sigma < 0$, and c, we have $\|Hx - cx;T,\sigma\| \leq r\|x;T,\sigma\|$. H is <u>incrementally exterior conic (c,r) with weight σ</u> if the preceding condition holds with the inequality reversed. H is <u>incrementally positive with weight σ</u> if for some constant $\sigma < 0$ we have $\langle(x - y), (Hx - Hy);T,\sigma\rangle \geq 0$. Again, if a given incremental condition is satisfied by a relation then so is the corresponding non-incremental condition (for the same weight).

<u>Remark 1</u> If any one of these incremental or nonincremental conditions is satisified for a given weight $\sigma_1 < 0$, then (as is easily shown) it is also satisfied for any weight σ with $\sigma_1 \leq \sigma < 0$.

Of special interest are those system elements which have a single input and a single output (so that the underlying function space has elements x which map R into itself rather than into R^m). In such cases, we can single out certain classes of operators (a special case of the usual representation of a system element by a relation) for which the concepts of gain, conicity, sectoricity, and positivity have particularly simple analytical (and, often, graphical) interpretations. Definitions 1, 2, and 3 delineate three such classes for the case of continuous time systems.

Definition 1 Let $\mathcal{N}$ denote the class of operators on X_e having the following properties: If $N \in \mathcal{N}$, then there is a function $N: R \twoheadrightarrow R$ such that $Nx(t) = N(x(t))$ for all $x \in X_e$, $t \in R$, where $N(\cdot)$ has the properties that $N(0) = 0$ and there exists a constant F such that $|N(y)| \leq F|y|$ for all $y \in R$.

Such an operator is causal, memoryless, and time-invariant, but not necessarily linear, and may be characterized by a graph in the instantaneous input-output plane. Necessarily, $g_{L_2}(N) \leq F$ and $g_{L_\infty}(N) \leq F$.

Remark 2 If $N \in \mathcal{N}$ is an operator on L_{2e} and if $N(\cdot)$ is the corresponding function, then:

(i) N is interior conic (c,r) if $|N(x) - cx| \leq r|x|$ for all $x \in R$.

(ii) N is exterior conic (c,r) if $|N(x) - cx| \geq r|x|$ for all $x \in R$.

(iii) N is positive if $xN(x) \geq 0$ for all $x \in R$.

Similarly, if N is an operator on $L_{\infty e}$, then the same conditions imply that N is interior conic (c,r) with weight σ, etc., for *any* $\sigma < 0$. The corresponding incremental conditions should be obvious.

Definition 2   Let $\mathcal{L}$ denote the class of operators on X_e having the following properties: If $H \in \mathcal{L}$, then there exists a function $h: R \rightarrow R$ and two sequences $\{h_i, i \in I^+\}$ and $\{t_i, i \in I^+\}$ such that

$$Hx(t) = \sum_{i \in I^+} h_i \, x(t-t_i) + \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

where

(i) $h(\cdot)$ has the properties that $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$ and $h(t) = 0$ for $t < 0$,

(ii) $\{t_i, i \in I^+\}$ is a sequence in R^+ with $t_i \leq t_{i+1}$ for each $i \in I^+$ and $\{h_i, i \in I^+\}$ is such that $\sum_{i \in I^+} |h_i| < \infty$.

Such an operator is causal, time-invariant, and linear, but not memoryless, and may be characterized by the corresponding Laplace transform

$$H(s) = \sum_{i \in I^+} h_i \exp(-st_i) + \int_0^{\infty} h(t) \exp(-st)\, dt$$

This representation is guaranteed to converge for Re $s \geq 0$ and may be extended to other complex values of s by analytic continuation. For the special case $s = j\omega$, $\omega \in R$, we have

$H(j\omega)$ which is termed (somewhat loosely) the <u>Fourier transform</u> or <u>frequency response</u> of the operator H. The quantity $\overline{H}(j\omega)$ defined by Re $\overline{H}(j\omega)$ = Re $H(j\omega)$ and Im $\overline{H}(j\omega) = \omega$Im $H(j\omega)$ is termed the <u>modified frequency response</u> of H. The graph of $H(j\omega)$ in the complex plane for $\omega \in R^+$ is termed the <u>Nyquist plot</u> of H. It is well-known that

$$g_{L_2}(H) = \operatorname*{ess\,sup}_{\omega \in R^+} |H(j\omega)|$$

and

$$g_{L_\infty}(H) = \sum_{i \in I^+} |h_i| + \int_0^\infty |h(t)|\, dt.$$

If $x \in L_2$ has limit-in-the-mean Fourier transform $X(j\omega)$ and if $y = Hx$ with $H \in \mathcal{L}$, then $y \in L_2$ and has limit-in-the-mean Fourier transform $Y(j\omega) = H(j\omega)\, X(j\omega)$.

<u>Remark 3</u> If $H \in \mathcal{L}$ is an operator on L_{2e} with Fourier transform $H(j\omega)$, then:

(i) H is incrementally interior conic (c,r) provided that $|H(j\omega) - c| \leq r$ for all $\omega \in R^+$. Graphically, this requires that the Nyquist plot of H lie inside a circle in the complex plane of radius r centered at $c + j0$.

(ii) H is incrementally exterior conic (c,r) provided that

$|H(j\omega) - c| \geq r$ for all $\omega \in R^+$ (so that the Nyquist plot of H lies outside of a circle in the complex plane of radius r centered at c + j0) and that the Nyquist plot of H does not encircle the point c + j0.

(iii) H is incrementally positive if Re $H(j\omega) \geq 0$ for all $\omega \in R^+$, i.e., if the Nyquist plot of H lies entirely in the closed right-half plane.

Remark 4 The conditions of Remarks 2 and 3 can be generalized to the case of operators with multiple inputs and multiple outputs, but much of their utility is lost since the generalized conditions do not have such simple graphical interpretations. For example, if H is an operator on L_{2e}^n which is represented by a matrix of operators H_{ij} with each H_{ij} a member of class $\mathcal{L}$, then, letting $H(j\omega)$ represent the corresponding matrix of Fourier transforms, H is interior conic (c,r) if the matrix $\left[(H^*(j\omega)-cI)'(H(j\omega)-cI) - r^2 I\right]$ is negative semi-definite for all $\omega \in R^+$. (Here * denotes complex conjugation.)

Definition 3 Let $\mathcal{C}$ denote the class of operators on X_e with the following properties: If $H \in \mathcal{C}$, then $H \in \mathcal{L}$ and there exists a real constant $\mu_0 < 0$ such that

$$\sum_{i \in I^+} |h_i| e^{-\mu_0 t_i} < \infty$$

and

$$\int_0^{\infty} |h(t)| \, e^{-\mu_0 t} \, dt < \infty .$$

For such an operator, the representation of the Laplace transform of H given following Definition 2 converges uniformly for all Re $s \geq \mu_0$ and, therefore, defines an analytic function in that region. The number μ_0 will be called an <u>abscissa of convergence</u> of the operator H. If $\mu_0 \leq \mu < 0$, then the function $H(\mu + j\omega)$ is well-defined for $\omega \in R$ and will be called the <u>μ-shifted Fourier transform</u> of H. The corresponding graph in the complex plane, for $\omega \in R^+$, is called the <u>μ-shifted Nyquist plot</u> of H. (This terminology is due to Zames[6].)

<u>Remark 5</u> Let $H \in \mathcal{C}$ be an operator on $L_{\infty e}$ with abscissa of convergence $\mu_0 < 0$. Then it can be shown that:

(i) If $|H(\mu + j\omega) - c| \leq r$ for all $\omega \in R^+$ and some real constants $r \geq 0$, c, and μ with $\mu_0 \leq \mu < 0$, then H is incrementally interior conic (c,r) with weight μ. (This is equivalent to the requirement that the μ-shifted Nyquist plot of H lie inside a circle in the complex plane of radius r centered at c + j0.)

(ii) If $|H(\mu + j\omega) - c| \geq r$ for all $\omega \in R^+$ and some real constants $r \geq 0$, c, and μ with $\mu_0 \leq \mu < 0$ (so that the μ-

shifted Nyquist plot of H lies outside a circle in the complex plane of radius r centered at c + j0) and if the μ-shifted Nyquist plot of H does not encircle the point c + j0, then H is incrementally exterior conic (c,r) with weight μ.

(iii) If Re $H(\mu + j\omega) \geq 0$ for all $\omega \in R^+$ (so that the μ-shifted Nyquist plot of H does not intersect the right-half plane) and some μ with $\mu_0 \leq \mu < 0$, then H is incrementally positive with weight μ.

Finally, we define a smoothing condition first introduced by Zames (see Reference 6):

Definition 4 A relation H has decaying L_1-memory if there exists an absolutely integrable, nonnegative, nonincreasing real-valued function m(t) defined for $t \in R^+$ such that

$$|He(t)|^2 \leq \int_{-\infty}^{t} |e(\tau)|^2 \, m(t-\tau) \, d\tau$$

for all e in the domain of H.

Remark 6 If $H \in \mathcal{C}$ with abscissa of convergence $\mu_0 < 0$ is such that $h_i = 0$ for each $i \in I^+$ and that

$$\int_0^{\infty} |h(t)|^2 \; e^{-2\mu_0 t} \, dt < \infty$$

then H has decaying L_1-memory. In order to verify this statement, let σ be such that $\mu_0 \leq \sigma < 0$. Then, using the definition of H and the Schwarz inequality (for L_2) we have

$$|He(t)|^2 = \left[\int_{-\infty}^{t} e(\tau)\, h(t-\tau)\, d\tau\right]^2$$

$$\leq \int_{-\infty}^{t} e^2(\tau) \exp\left[2\sigma(t-\tau)\right] d\tau \cdot \int_{-\infty}^{t} h^2(t-\tau) \exp\left[-2\sigma(t-\tau)\right] d\tau$$

$$= \int_{-\infty}^{t} |e(\tau)|^2\, m(t-\tau)\, d\tau$$

where $m(t) = m(0) \exp(2\sigma t)$ and $m(0) = \int_{0}^{\infty} h^2(t) \exp(-2\sigma t)\, dt$.

Remark 7 If H has decaying L_1-memory with memory function $m(t)$ and if the input to H (call it e) satisfies an inequality of the form

$$\exp(2\mu t)\, \|e;t,\mu\|^2 \leq (a + b\, \|x\|_{L_\infty})^2$$

for all $t \in R$ and for some constants $a, b \in R^+$, then we have

$$|He(t)|^2 \leq (a + b\, \|x\|_{L_\infty})^2\, K^2$$

where K is a positive constant with

$$K^2 = \int_{0}^{\infty} |m(t)|\, dt + (-2\mu)^{-1}\left[m(0) + \int_{-\infty}^{0} \frac{dm(-t)}{dt}\, dt\right]$$

This in turn implies that

$$\|He\|_{L_\infty} \leq (a + b \|x\|_{L_\infty}) K.$$

Remarks 6 and 7 play an important role in Chapter Six. For a proof of the claims of Remark 7, see Reference 6.

Discrete Time Systems

Let x be an arbitrary function which maps I into R^m. Thus, x is characterized by a sequence of m-vectors $\{x(i), i \in I\}$ with $x(i) \in R^m$ for each i. In particular, we are interested in sequences such that each x(i) depends only on a corresponding time t_i where $\{t_i, i \in I\}$ is a sequence of times with $t_0 = 0$ and $t_i \leq t_{i+1}$ for each $i \in I$. Defining two symbols

$$\|x\|_{l_2} = \left[\sum_{i \in I} |x_i|^2\right]^{\frac{1}{2}}$$

and

$$\|x\|_{l_\infty} = \sup_{i \in I} |x_i|,$$

the normed spaces $l_2(I)$ and $l_\infty(I)$ (hereafter denoted by l_2 and l_∞, respectively) are defined by

$$l_2 = \left\{ x : I \longrightarrow R^m \,\middle|\, \|x\|_{l_2} < \infty \right\}$$

and

$$l_\infty = \left\{ x : I \longrightarrow R^m \,\middle|\, \|x\|_{l_\infty} < \infty \right\}$$

If $x, y \in l_2$, then the inner product of x and y is defined by

$$\langle x, y \rangle_{l_2} = \sum_{i \in I} x_i \cdot y_i .$$

The truncation of x at time t_N, denoted by x_N, is the sequence defined by

$$x_N(i) = \begin{cases} x(i) & \text{for } t_i \leq t_N \text{ (i.e., } i \leq N) \\ 0 & \text{for } t_i > t_N \text{ (i.e., } i > N) \end{cases}$$

As usual, we introduce the corresponding extended spaces

$$l_{2e} = \left\{ x : I \longrightarrow R^m \,\middle|\, x_N \in l_2 \text{ for all } N \in I \right\}$$

$$l_{\infty e} = \left\{ x : I \longrightarrow R^m \,\middle|\, x_N \in l_\infty \text{ for all } N \in I \right\}$$

For the remainder of this Chapter, let X_e stand for either l_{2e} or $l_{\infty e}$. A multiple-input multiple-output discrete time system is modeled as a relation on the product space X_e^n. Thus, inputs and outputs are assumed to belong to X_e. System elements are represented by relations on X_e. For such a system, stability is interpreted as boundedness and continuity of the relations which connect system inputs to each system output.

A relation H on X_e is characterized by a (doubly-infinite) array $\{H(ij);\ i,j \in I\}$ of relations H(ij) having a domain which is a subset of R and a range which is a subset of R^m. Thus, if y = Hx, we have $y_i = \sum_{j \in I} H(ij)(x(j))$ for each $i \in I$. Such a relation is causal if H(ij) = 0 for $j > i$, time-invariant if H(ij) depends only on the value of i-j, and memoryless if H(ij) = 0 whenever $i \neq j$.

First, let us consider multiple-input multiple-output discrete time systems for which the underlying extended function space is l_{2e}. In order to discuss the boundedness of such systems, we introduce a number of definitions analogous to those introduced for the continuous time case. In each of the following definitions, H is a relation on l_{2e} and the indicated condition is understood to apply for all $N \in I$, $x \in Do(H)$, and $Hx \in Ra(H)$. The <u>l_2-gain</u> of H, $g_{l_2}(H)$, is the smallest nonnegative number M such that $\|(Hx)_N\|_{l_2} \leq M\|x_N\|_{l_2}$. H is <u>interior conic (c,r)</u> if there are real numbers $r \geq 0$, and c such that $\|(Hx)_N - cx_N\|_{l_2} \leq r\|x_N\|_{l_2}$. H is <u>exterior conic (c,r)</u> if the preceding condition holds with the inequality reversed. H is <u>inside the sector $\{a,b\}$</u> if for some real numbers a,b we have $\langle (Hx)_N - ax_N, (Hx)_N - bx_N \rangle_{l_2} \leq 0$. H is <u>outside the sector $\{a,b\}$</u> if the preceding condition holds with the inequality reversed. H is <u>positive</u> if $\langle x_N, (Hx)_N \rangle_{l_2} \geq 0$.

In order to discuss the continuity of such discrete time systems, we introduce six corresponding incremental conditions. Since these can be obtained from the corresponding incremental definitions for the continuous time case in an obvious way, they shall not be given explicitly.

Next, consider multiple-input multiple-output discrete time systems in which the underlying extended function space is $l_{\infty e}$. Let H be a relation on $l_{\infty e}$. The <u>l_∞-gain</u> of H, $g_{l_\infty}(H)$, is the smallest nonnegative number M such that $\|(Hx)_N\| \leq M\|x_N\|$. (This and later conditions are understood to apply for all $N \in I$, $x \in Do(H)$, and $Hx \in Ra(H)$.) In analogy to the procedure for continuous time systems, we convert elements of $l_{\infty e}$ into elements of l_{2e} by supplying an appropriate exponential weight. Denote by e^{σ} the exponential function which takes on the value $e^{\sigma i}$ at time t_i. (Note that the weighting is in the index or i-space and is not a time weighting in general unless the time instants t_i are evenly spaced.) Then, if $x \in l_{\infty e}$, we have $x_N e^{-\sigma} \in l_2$ for any $\sigma < 0$.

We introduce symbols for the norm and inner product of truncated, weighted sequences ($\sigma < 0$ is understood):

$$\|x;N,\sigma\| = \|x_N e^{-\sigma}\|_{l_2}$$

$$\langle x,y;N,\sigma\rangle = \langle x_N e^{-\sigma}, y_N e^{-\sigma}\rangle_{l_2}.$$

The conditions for a relation H on $l_{\infty e}$ to be interior conic (c,r) with weight σ, exterior conic (c,r) with weight σ, and positive with weight σ are obtained from the corresponding conditions for the continuous time case by replacing $L_{\infty e}$ and R with $l_{\infty e}$ and I, respectively. A similar statement applies to the corresponding incremental conditions. Remark 1 also applies to the discrete time case.

Of special interest are those systems containing system elements having a single input and a single output (so that the underlying function space has elements x which map I into R rather than R^m). In such cases, we can single out certain classes of operators (just as we did for the continuous time case) for which the concepts of gain, conicity, sectoricity, and positivity have simple analytical or graphical interpretations. Recall that, in the present section, X_e stands for either l_{2e} or $l_{\infty e}$.

Definition 5   Let $\mathcal{M}$ denote the class of operators on X_e with the following properties: If $N \in \mathcal{M}$, then there is a function $N: R \longrightarrow R$ such that $(Nx)(i) = N(x(i))$ for all $x \in X_e, i \in I$, where $N(\cdot)$ has the properties that $N(0) = 0$ and there exists a constant F such that $|N(y)| \leqslant F|y|$ for all $y \in R$.

Such an operator is causal, memoryless, and time-invariant, and may be characterized by a graph in the instantaneous input-output plane. Necessarily, $g_{l_\infty}(N) \leqslant F$ and $g_{l_2}(F) \leqslant N$.

Remark 8   If $N \in \mathcal{M}$ is an operator on l_{2e} and if $N(\cdot)$ is the corresponding function, then:

(i) N is interior conic (c,r) if $|N(x) - cx| \leqslant r|x|$ for all $x \in R$.

(ii) N is exterior conic (c,r) if $|N(x) - cx| \geqslant r|x|$ for all $x \in R$.

(iii) N is positive if $xN(x) \geqslant 0$ for all $x \in R$.

If $N \in \mathcal{M}$ is an operator on $l_{\infty e}$, then the same conditions imply that N is interior conic (c,r) with weight σ, etc., for any $\sigma < 0$. The corresponding incremental conditions should be obvious.

Definition 6   Let $\mathcal{L}$ denote the class of operators on X_e having the following properties: If $H \in \mathcal{L}$, then there exists a sequence of real numbers $\{h(k), k \in I^+\}$ such that

$$(Hx)(i) = \sum_{k \in I^+} h(k)\ x(i-k)$$

for each $i \in I$ and such that

$$\sum_{k \in I^+} |h(k)| < \infty .$$

Such an operator is causal, time-invariant, and linear, and may be characterized by the corresponding transform

$$H(z) = \sum_{k \in I^+} h_k\ z^{-k}$$

which converges (at least) for complex z with $|z| \geq 1$. For z with $|z| = 1$, this transform (and the corresponding graph in the complex plane) is called the <u>z-transform</u> of H. It is well-known that

$$g_{l_2}(H) = \max_{|z| = 1} |H(z)|$$

and

$$g_{l_\infty}(H) = \sum_{i \in I^+} |h_i| .$$

If $x \in l_2$ has limit-in-the-mean z-transform X(z) and if

$H \in \mathcal{Q}$ is an operator on l_{2e}, then $y = Hx$ belongs to l_2 and has limit-in-the-mean z-transform $Y(z) = H(z)\,X(z)$ for $|z| = 1$.

Remark 9   Only if the underlying time instants t_i are evenly spaced (so that $t_i = iP$ for each $i \in I$ for some real number P called the sampling period) does this z-transform have the conventional frequency domain interpretation. That is, if the system in question is the (ideally) sampled version of a linear continuous time system characterized by a Laplace transform H(s), then only in this special case can we obtain the corresponding z-transform, denoted by H(z), by the simple transformation $z = \exp(sP)$.

Remark 10   If $H \in \mathcal{Q}$ is an operator on l_{2e} and has z-transform H(z), then:

(i) H is incrementally interior conic (c,r) if $|H(z) - c| \leqslant r$ for all $|z| = 1$.

(ii) H is incrementally exterior conic (c,r) if $|H(z) - c| \geqslant r$ for all $|z| = 1$ and if the plot of H(z) does not encircle the point $c + j\,0$.

(iii) H is incrementally positive if $\operatorname{Re} H(z) \geqslant 0$ for all $|z| = 1$.

Definition 7   Let $\mathcal{C}$ denote the class of operators on

X_e having the following properties: If $H \in \mathcal{O}$, then $H \in \mathcal{l}$ and there exists a constant $\mu_o < 0$ such that the sequence $\{h_i e^{-\mu_o i}, i \in I^+\}$ is absolutely summable, i.e., such that

$$\sum_{i \in I^+} |h_i| \, e^{-\mu_o i} < \infty$$

The constant μ_o shall be termed an abscissa of convergence of the operator H. If $\mu_o \leqslant \mu < 0$, then the representation of H(z) given following Definition 6 converges (at least) for $|z| > e^{\mu}$. Therefore, $H(z\, e^{\mu})$ converges for $|z| = 1$ and will be called the <u>μ-shifted z-transform</u> of H. (Although a rather straightforward extension of the corresponding continuous time quantity introduced by Zames[6], the μ-shifted z-transform has, apparently, never been considered in the literature before.)

<u>Remark 11</u> If $H \in \mathcal{O}$ is an operator on $l_{\infty e}$ with abscissa of convergence μ_o and if $\mu_o \leq \mu < 0$, then the positivity and conicity conditions of Remark 5 apply provided that $H(\mu + j\omega)$ is replaced by $H(z\, e^{\mu})$ and that the restriction $\omega \in R^+$ is is replaced by the restriction $|z| = 1$. The graphical interpretations of the various conditions is unchanged.

<u>Definition 8</u> A relation H has <u>decaying l_1-memory</u> if there exists an absolutely summable, nonincreasing, nonnegative function $m: I^+ \longrightarrow R$ such that

$$\left|(Hx)(i)\right|^2 \leqslant \sum_{k=-\infty}^{i} \left|e(k)\right|^2 m(i-k).$$

Remark 12 If $H \in \mathcal{C}$ with abscissa of convergence $\mu_o < 0$ is such that

$$\sum_{i \in I^+} \left|h(i)\right|^2 e^{-2\mu_o i} < \infty ,$$

then H has decaying l_1-memory. This statement is easily verified by an argument closely analogous to the one presented in Remark 6 for the continuous time case.

Remark 13 If H has decaying l_1-memory with memory function $m(\cdot)$ and if the input to H (call it e) satisfies an inequality of the form

$$J(i) = \exp(2\mu i) \left\|e;i,\mu\right\|^2 \leqslant (a + b \left\|x\right\|_{1_\infty})^2$$

for all $i \in I$ and some real constants $a, b \in R^+$, then we have

$$\left\|He\right\|_{1_\infty} \leqslant (a + b \left\|x\right\|_{1_\infty}) K$$

where K is a positive constant. To see this, first note that

$$\left|e(i)\right|^2 = J(i) - e^{2\mu} J(i-1).$$

Then, using the definition of decaying l_1-memory, we have

$$\left|(He)(n)\right|^2 \leq \sum_{i=-\infty}^{n} m(n-i) \left[J(i) - e^{2\mu} J(i-1)\right]$$

$$= \sum_{i=-\infty}^{n} m(n-i)\, e^{2\mu i} \left[\|e;i,\mu\|^2 - \|e;i-1,\mu\|^2\right]$$

$$\leq \sum_{i=-\infty}^{n} m(n-i)\, e^{2\mu i}\; 2\,\|e;i,\mu\|^2$$

$$\leq 2\,(a + b\,\|x\|_{1\infty})^2 \sum_{k\in I^+} m(k).$$

Defining $K^2 = 2 \sum_{k\in I^+} m(k)$ and taking the supremum of the resulting inequality over all $n \in I$, we have the desired result.

Although not every class of operators defined in this Chapter shall be referred to explicitly in what follows, the above characterizations should aid the reader in identifying systems to which the results of this paper can be successfully applied.

CHAPTER THREE: SINGLE-LOOP SYSTEMS

Stability Conditions

The theorems and definitions presented in this section apply to both continuous time and discrete time systems. For the sake of brevity, these theorems and definitions will be phrased in terms of the spaces X and X_e and in terms of the time interval of definition S. These symbols are to be interpreted as L_2, L_{2e}, and R, respectively, if the system in question is a continuous time system, and as l_2, l_{2e}, and I if it is a discrete time one. In this same spirit, all abstract functional equations given in this section are to be interpreted as governing either discrete or continuous time systems as the occasion demands.

Let M be a relation on the product space X_e^n which is represented by an $n \times n$ matrix with components M_{ij}, each of which is itself a relation on X_e. Then, if $x \in X_e^n$ and if $y = Mx$, we have

$$y_i = \sum_{j=1}^{n} M_{ij} x_j.$$

Therefore, for each $T \in S$, we have

$$\|y_{iT}\| \leq \sum_{j=1}^{n} \|(M_{ij}x_j)_T\| \leq \sum_{j=1}^{n} g(M_{ij}) \|x_{jT}\|$$

Introducing the notation

$$X_T = (\|x_{1T}\|, \|x_{2T}\|, \dots, \|x_{nT}\|)'$$

$$Y_T = (\|y_{1T}\|, \|y_{2T}\|, \dots, \|y_{nT}\|)' \tag{1}$$

$$G(M) = \left[g(M_{ij})\right]$$

this result becomes $Y_T \leq G(M)\, X_T$. Therefore, we also have

$$Y_T' \cdot Y_T \leq X_T' G(M)' G(M) X_T. \tag{2}$$

It is easy to show that

$$\|y\|_n = \sqrt{Y' \cdot Y} = \sqrt{\sum_{i=1}^{n} \|y_i\|^2}$$

is a norm on the product space X_e^n. Using (1), we have

$$\|y_T\|_n^2 = Y_T' \cdot Y_T \leq X_T' G(M)' G(M) X_T$$

$$\leq E(G(M)'G(M))\; X_T' \cdot X_T \leq E(G(M)'G(M)) \|x_T\|_n$$

for all $T \in S$, where $E(G(M)'G(M))$ is the largest eigenvalue of the real symmetric matrix $G(M)'G(M)$. Clearly, the positive square root of $E(G(M)'G(M))$ provides an upper bound on the gain of the matrix mapping M.

Consider the multiple-input multiple-output single-loop feedback system depicted in Figure 1 and governed by the abstract functional equations

$$
\begin{aligned}
y_i &= \sum_{j=1}^{n} H_{ij}e_j, \quad e_i = x_i + w_i + z_i, \\
z_i &= \sum_{j=1}^{n} B_{ij}f_j, \quad f_i = u_i + v_i + y_i,
\end{aligned}
\tag{3}
$$

for $i = 1,2,\ldots n$. Here, w_i and v_i are reference signals and belong to X, while the inputs x_i, u_i, the error signals e_i, f_i, and the outputs y_i, z_i, are all assumed to belong to X_e. Each B_{ij} and H_{ij} is assumed to be a relation on X_e. Associated with system (3) are the relations E_{ij}, F_{ij}, Y_{ij}, and Z_{ij} which connect the inputs x_j, u_j with the errors e_i, f_i or the outputs y_i, z_i (the reference signals are taken to be fixed). According to the work of Zames[4-5], this system will be input-output bounded provided that the open-loop gain product is less than one. Using the matrix gain just developed, this will be the case if either $E(G(BH)'G(BH)) < 1$ or $E(G(HB)'G(HB)) < 1$. (Clearly, these conditions are satisfied, respectively, if either $E(G(H)'G(B)'G(B)G(H)) < 1$ or

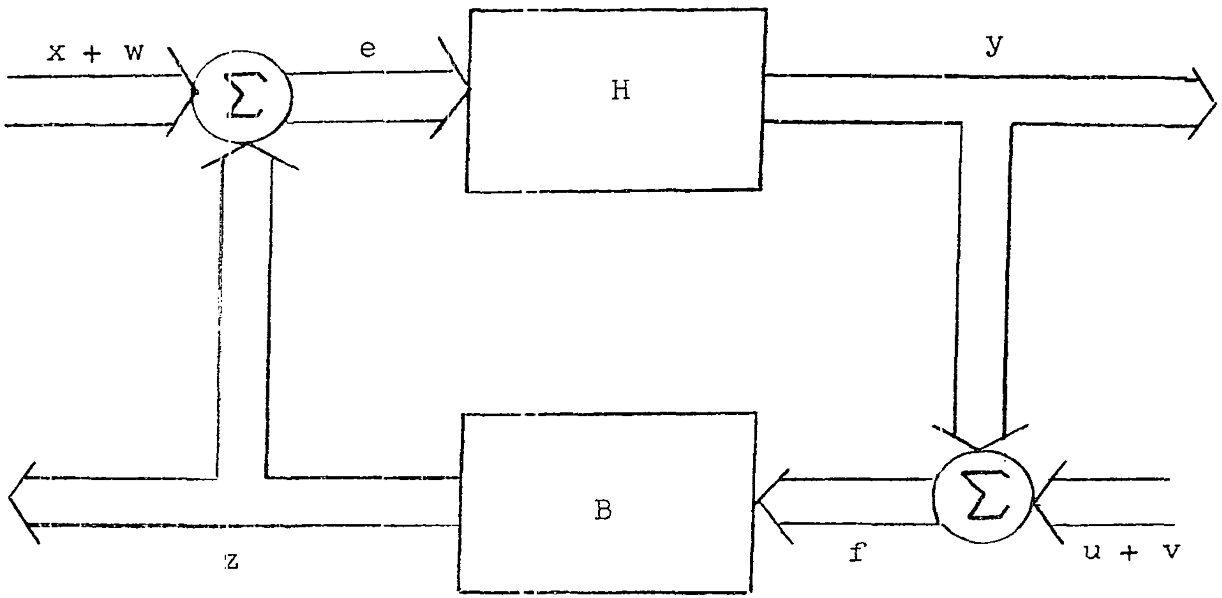


Figure 1: Multiple-input multiple-output single-loop system (3).

$E(G(B)'G(H)'G(H)G(B)) < 1.$) Here, BH and HB are the open-loop matrix relations and G(BH) and G(HB) are the corresponding matrices of gains. Similar conditions may be given for the continuity of the closed-loop system (3) in terms of the incremental open-loop gains.

Our goal in this section is to obtain a different (and, we hope, more readily applicable) boundedness condition (and corresponding continuity condition) for the system of Figure 1. This new condition is a generalization of the one presented in Reference 34. From Equations (3), we easily derive

$$e_i = x_i + w_i + \sum_{j=1}^{n} B_{ij}(u_j + v_j + \sum_{k=1}^{n} H_{jk}e_k) \tag{4}$$

Truncating this equation at time $T \in S$, using the triangle inequality, and the definition of gain, we obtain

$$\|e_{iT}\| \leq \|x_{iT}\| + \|w_{iT}\| + \sum_{j=1}^{n} g(B_{ij}) \Big[\|u_{jT}\| + \|v_{jT}\| + \sum_{k=1}^{n} g(H_{jk}) \|e_{kT}\| \Big]$$

so that

$$\|e_{iT}\| - \sum_{k=1}^{n} \sum_{j=1}^{n} g(B_{ij})\, g(H_{jk})\, \|e_{kT}\|$$

$$\leqslant \|x_{iT}\| + \|w_{iT}\| + \sum_{j=1}^{n} g(B_{ij}) \left[\|u_{jT}\| + \|v_{jT}\| \right].$$

Defining (time-truncated) column vectors E_T, X_T, W_T, U_T, and V_T and matrices of gains $G(B)$ and $G(H)$ in analogy to Equations (1), this may be re-expressed as

$$\left[I - G(B)G(H) \right] E_T \leqslant X_T + W_T + G(B) \left[U_T + V_T \right].$$

If the successive principal minors of the test matrix $T = I - G(B)G(H)$ are all positive, then T^{-1} exists and all its components are nonnegative. (This result depends on the special form of this test matrix. For a proof, see Reference 34.) If, in addition, all components of $G(B)$ are finite, then there exist nonnegative numbers δ_{ij} and ρ_{ij} for $i,j = 1,2,\ldots,n$ which are independent of T such that

$$\|e_{iT}\| \leqslant \sum_{j=1}^{n} \delta_{ij} \left(\|x_{jT}\| + \|w_{jT}\|\right) + \sum_{j=1}^{n} \rho_{ij} \left(\|u_{jT}\| + \|v_{jT}\|\right)$$

Note that $\|e_{iT}\|$, $\|x_{iT}\|$, $\|w_{iT}\|$, $\|u_{iT}\|$, and $\|v_{iT}\|$ are monotone increasing functions of T and that w_i and v_i are assumed to belong to X for each $i = 1,2,\ldots,n$. Restricting each x_j and u_j to lie in a bounded subset of X_e and letting T approach infinity first on the right-hand side and then on the left-hand side of this inequality, we obtain

$$\|e_i\| \leqslant \sum_{j=1}^{n} \delta_{ij} \left(\|x_j\| + \|w_j\|\right) + \sum_{j=1}^{n} \rho_{ij} \left(\|u_j\| + \|v_j\|\right).$$

But, this means that each relation E_{ij} mapping inputs into errors e_i is bounded. If, in addition, each gain $g(H_{ij})$ is finite, then it is easy to see that the relations Y_{ij} are bounded as well. Boundedness of the relations Z_{ij} and F_{ij} follows from that of E_{ij} and Y_{ij}, respectively. Therefore, we have proved the following result:

Theorem 1 If each gain $g(H_{ij})$ and $g(B_{ij})$ is finite, then the relations E_{ij}, F_{ij}, Y_{ij}, and Z_{ij} associated with the multiple-input multiple-output system of Equations (3) are bounded provided that the successive principal minors of the test matrix $T = I - G(B)G(H)$ are all positive.

Remark 14 This Theorem reproduces the result given in Reference 34 in the special case in which the matrix H of forward loop relations is diagonal and in which all feedback relations are just constant multipliers, i.e., $B_{ij} = b_{ij}$ for each $i,j = 1,2,\ldots,n$.

By interchanging the roles of B and H in the previous development, we obtain the result:

Corollary Theorem 1 holds if the test matrix $I - G(B)G(H)$ is replaced by the test matrix $I - G(H)G(B)$.

<u>Remark 15</u> If some, or all, of the gains $g(B_{ij})$ and $g(H_{ij})$ are replaced by larger numbers (such as upper bounds on these gains), then positivity of the successive principal minors of the resulting test matrix is still sufficient to guarantee boundedness. Similar remarks apply to all Theorems given in this paper.

It can be shown (see Reference 34) that the positivity conditions of Theorem 1 imply that the (possibly complex) eigenvalues of the real matrix $G(B)G(H)$ are less than one in absolute value. This is to be compared with the more conventional condition which requires that the largest eigenvalue (all of which are real) of the real symmetric matrix $G(B)'G(H)'G(H)G(B)$ be less than one. Although superficially, quite similar, these boundedness conditions are not equivalent, nor is one a special case of the other. The conditions of Theorem 1 have the advantage that they are much easier to apply to the systems considered later in this paper. (They lead to conditions which are linear in each margin of boundedness parameter, whereas the corresponding conventional conditions lead to constraints quadratic in each margin of boundedness parameter.)

Similar conditions can be derived to ensure the continuity of system (3). The starting point of the derivation is to consider the equation obtained by subtracting equation (4) for one (arbitrary) choice of the inputs x_i and u_i from the same equation for a second such arbitrary choice. Repeating the arguments given earlier (with gains replaced by the corresponding incremental gains) leads to the following:

<u>Theorem 2</u> If each incremental gain $g^I(H_{ij})$ and $g^I(B_{ij})$ is finite, then the relations E_{ij}, F_{ij}, Y_{ij}, and Z_{ij} associated with the multiple-input multiple-output system (3) are <u>continuous</u> provided that the successive principal minors of the test matrix $T = I - G^I(B)G^I(H)$ are all positive. A similar statement holds with the test matrix T replaced by $I - G^I(H)G^I(B)$.

<u>Remark 16</u> If a relation is continuous, it is neccessarily bounded, so Theorem 2 provides sufficient conditions for both boundedness and continuity and, thus, for input-output stability (here, either L_2- or l_2-stability depending on whether the system in question is continuous time or discrete time).

Degree of Boundedness and Continuity

In this section, we define quantities which provide a measure of the degree of boundedness or of continuity of single-loop feedback systems of the form shown in Figure 1.

First, consider the case of a system described by equations (3) for which the underlying function space is L_2 or l_2 (so we are concerned with L_2- or l_2-stability). A quantity which provides a measure of the degree of boundedness of such a system is defined as follows:

<u>Definition 9</u> The single-loop system (3) has <u>margin of boundedness</u> δ if for some $0 < \delta < 1$, some $r \geq 0$, and some real constant c, we have B interior conic $(-c,(1-\delta)r)$ and one of the following conditions is satisfied:

(i) $c^2 > r^2$ and H is exterior conic $(-c/(c^2-r^2), r/(c^2-r^2))$ (so that H is outside the sector $\{-1/(c-r), -1/(c+r)\}$).

(ii) $r^2 > c^2$ and H is interior conic $(c/(r^2-c^2), r/(r^2-c^2))$ (so that H is inside the sector $\{-1/(c-r), -1/(c+r)\}$).

(iii) $c^2 = r^2$ and $2cH + I$ is positive.

We are particularly interested in systems which have a margin of boundedness δ and for which either $x = w = 0$ or $u = v = 0$. For such systems, we make an additional definition:

<u>Definition 10</u> If the single-loop system (3) has margin of boundedness δ, then the quantity μ will be called the <u>gain factor</u> of the system, where $\mu = r^{-1}$ in the special case $u = v = 0$ and $\mu = (|c| + r^{-1})(1 + |c|\, r^{-1})$ in the special

case $x = w = 0$.

The motivation for introducing the concept of gain factor is provided by the following observation: if a system has margin of boundedness δ and gain factor μ , then the quantity μ/δ is an upper bound on the overall input-output gain of the system. (This observation is justified by the calculations presented in Appendix A.)

Correspondingly, we introduce a quantity which provides a measure of the degree of continuity of system (3):

<u>Definition 11</u> The single-loop system (3) has <u>margin of continuity</u> δ_c if for some $0 < \delta_c < 1$, some $r \geq 0$, and some real constant c, we have B incrementally interior conic $(-c,(1-\delta_c)r)$ and one of the following conditions is satisfied:
(i) $c^2 > r^2$ and H is incrementally exterior conic $(-c/(c^2-r^2), r/(c^2-r^2))$ (so that H is incrementally outside the sector $\{-1/(c-r),-1/(c+r)\}$).
(ii) $r^2 > c^2$ and H is incrementally interior conic $(c/(r^2-c^2), r/(r^2-c^2))$ (so that H is incrementally inside the sector $\{-1/(c-r),-1/(c+r)\}$).
(iii) $c^2 = r^2$ and $2cH + I$ is incrementally positive.

For systems for which either $u = v = 0$ or $x = w = 0$, we define an <u>incremental gain factor</u> μ^I exactly as in the definition of gain factor except that the constant δ is replaced by δ_c and the constants c and r are the incremental constants of Definition 11. In these special cases, the quantity μ^I/δ_c provides an upper bound on the overall input-output incremental gain of the system.

Now, let us briefly consider the case of a system described by equations (3) for which the underlying function space is L_∞ or l_∞ (so that we are concerned with L_∞- or l_∞-stability). As before, we introduce a measure of the degree of boundedness of such a system:

<u>Definition 12</u> Single-loop system (3) (regarded as a relation on the product space L^n_∞ or l^n_∞) has <u>margin of boundedness</u> δ if for some $0 < \delta < 1$, some $r \geq 0$, and some real c, B is interior conic $(-c,(1-\delta)r)$ with weight $\sigma_2 < 0$ and one of the following holds:

(i) $c^2 > r^2$ and B is exterior conic $(-c/(c^2-r^2), r/(c^2-r^2))$ with weight $\sigma_1 < 0$.

(ii) $r^2 > c^2$ and H is interior conic $(c/(r^2-c^2), r/(r^2-c^2))$ with weight $\sigma_1 < 0$.

(iii) $r^2 = c^2$ and $2cH + I$ is positive with weight $\sigma_1 < 0$.

<u>Remark 17</u> If system (3) has margin of boundedness δ and if we define $\mu = \max\{\sigma_1, \sigma_2\}$, then we have (cf.,

Remark 1)

$$\|Bf + cf;t,\mu\| \leq (1-\delta)r\|f;t,\mu\| \tag{5}$$

For the sake of brevity, replace x + w by x and u + v by u in the remainder of this Remark. It is not hard to show that, with the given assumptions, the inequality

$$r\|He;t,\mu\| \leq \|cHe + e;t,\mu\| \tag{6}$$

is satisfied for all three cases of Definition 12. (The necessary algebra is given in Appendix A.) Using the equation e = x + Bf to eliminate e and f = u + He to eliminate He, we can rewrite (6):

$$r\|f - u;t,\mu\| \leq \|c(f-u) + x + Bf;t,\mu\|$$

This implies that

$$r\left[\|f;t,\mu\| - \|u;t,\mu\|\right] \leq \|x - cu;t,\mu\| + \|Bf + cf;t,\mu\|$$

$$\leq \|x - cu;t,\mu\| + (1-\delta)r\|f;t,\mu\|$$

where (5) has been used in the last step. Therefore, for all three cases of Definition 12, we have

$$\delta r \, \|f;t,\mu\| \leqslant r \, \|u;t,\mu\| + \|x - cu;t,\mu\| \qquad (7)$$

Also, using the equation e = x + Bf, we have

$$\|e;t,\mu\| = \|x + Bf;t,\mu\|$$

$$\leqslant \|x - cf;t,\mu\| + \|Bf + cf;t,\mu\|$$

Using (5), this implies

$$\|e;t,\mu\| \leqslant \|x;t,\mu\| + (|c| + (1-\delta)r) \, \|f;t,\mu\| \quad (8)$$

Consider two special cases:

(i) Suppose that u = 0, then (7) and (8) imply

$$\|e;t,\mu\| \leqslant \|x;t,\mu\| + \frac{|c| + (1-\delta)r}{\delta r} \|x;t,\mu\|$$

$$= \frac{|c| + r}{\delta r} \, \|x;t,\mu\|$$

(ii) Suppose that x = 0, then (7) implies

$$\|f;t,\mu\| \leqslant \frac{|c| + r}{\delta r} \, \|u;t,\mu\|$$

In the special cases $u = 0$ or $x = 0$, the quantity $(|c| + r)/r$ plays a role similar to that of a gain factor since $(|c| + r)/\delta r$ is an upper bound on the overall gain between the system input and the error signal (either x and e or u and f). Clearly, we can define a <u>margin of continuity</u> for the present case simply by replacing all conditions in Definition 12 by their incremental counterparts. A calculation analogous to that presented in Remark 17 may then be carried out. In the indicated special cases, the quantity $(|c| + r)/r$ divided by the margin of continuity provides an upper bound on the overall incremental gain between system input and error signal.

CHAPTER FOUR: STABILITY OF GENERAL INTERCONNECTED SYSTEMS

In this Chapter, we adopt the interconnected systems viewpoint. The model we study is a slight generalization of the one treated in Reference 34. The new model is governed by the following set of abstract functional equations:

$$e_i = x_i + w_i + \sum_{j=1}^{p} B_{ij} y_j, \qquad y_i = H_i e_i, \tag{9}$$

for $i = 1,2,\ldots,p$. Here, each input x_i, error signal e_i, and output y_i is assumed to belong to an extended function space (either L_{2e} or l_{2e}) while the reference signals w_i are assumed to belong to either L_2 or l_2. According to our present viewpoint, the "forward loop" relations H_i are thought of as describing a set of p isolated subsystems which are interconnected by the "feedback" relations B_{ij}. Viewpoint aside, system (9) is clearly a special case of multiple-input multiple-output system (3). The stability of system (9) is interpreted in terms of the boundedness and continuity of the relations E_{ij}, which connect x_j with e_i, and the relations Y_{ij}, which connect x_j with y_i.

Due to the absence of inputs analogous to u and v in system (3), we are able to obtain stability conditions for system (9) which are somewhat more general than the ones obtained from a straightforward application of Theorems 1 and 2 to this system. The proofs of the following Theorems are sketched in Appendix B:

<u>Theorem 3</u> All relations E_{ij} associated with interconnected system (9) are <u>bounded</u> if all of the gains $g(B_{ij}H_j)$ are finite and the successive principal minors of the test matrix $T = I - \left[g(B_{ij}H_j)\right]$ are positive. If, in addition, each gain $g(H_j)$ is finite, then all relations Y_{ij} are bounded as well.

<u>Theorem 4</u> All relations E_{ij} associated with interconnected system (9) are <u>continuous</u> if all the incremental gains $g^I(B_{ij}H_j)$ are finite and the successive principal minors of the test matrix $T = I - \left[g^I(B_{ij}H_j)\right]$ are positive. If, in addition, each incremental gain $g^I(H_j)$ is finite, then all relations Y_{ij} are continuous as well.

In order to facilitate comparisons between various results, interconnected system (9) is depicted in block diagram form in Figure 2.

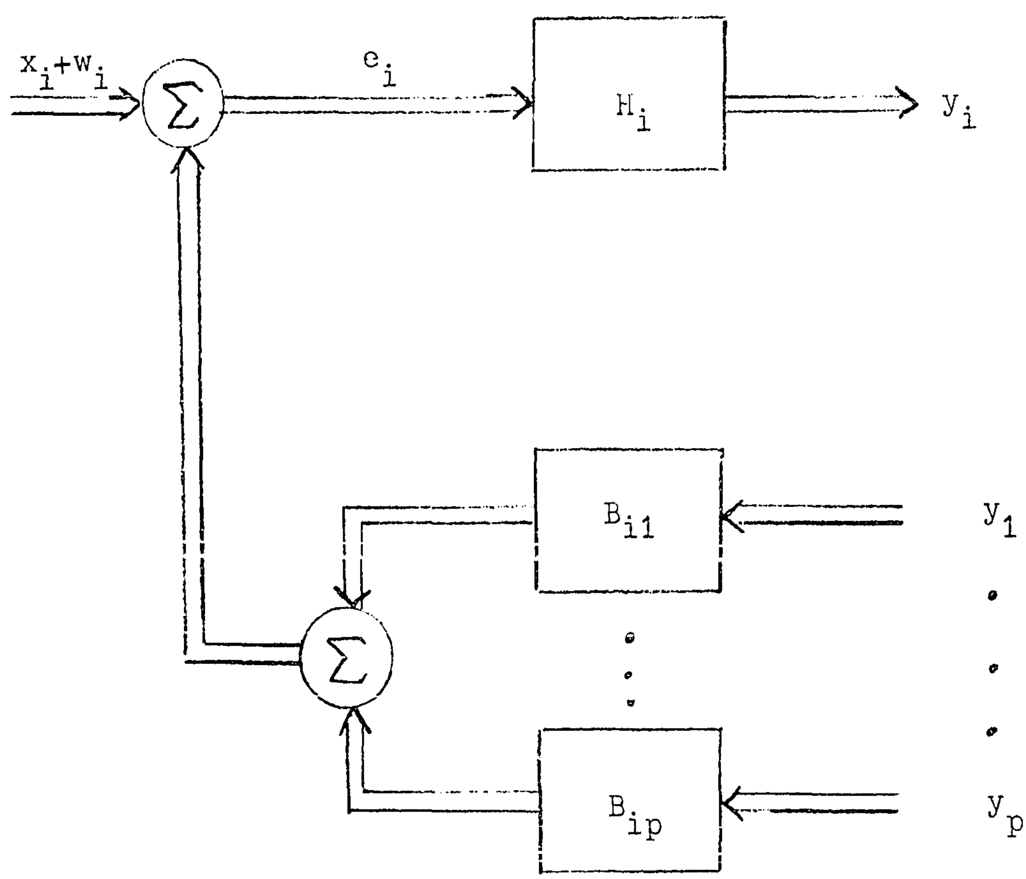


Figure 2: Interconnected system (9).

CHAPTER FIVE: INTERCONNECTED FEEDBACK SYSTEMS -- GENERAL RESULTS FOR L_2- AND l_2-BOUNDEDNESS AND CONTINUITY

We now turn our attention to the class of systems of primary interest, namely, those systems which can be viewed as an interconnection of single-loop feedback systems. Thus, we concern ourselves with systems having the structure shown in Figure 3 and governed by the set of abstract functional equations

$$e_i = u_i + B_i y_i, \quad y_i = H_i e_i,$$
$$u_i = x_i + w_i + \sum_{j=1}^{p} C_{ij} y_j, \tag{10}$$

for $i = 1,2,\ldots,p$. Here, the underlying function space X_e is one of the inner product spaces L_{2e} or l_{2e}. For each i, the input x_i, error u_i or e_i, and output y_i is assumed to belong to the product space X_e^n, while each reference signal w_i is assumed to belong to X^n. Each H_i, B_i, and C_{ij} is a matrix of relations with n x n component relations, each of which maps X_e into itself. X_e, in turn, is a space of vector-valued functions such that, if $x \in X_e$, then $x(t) \in R^m$ for each $t \in S$. Thus, at any given instant of time, a typical signal, say $u_i(t)$, is characterized by a set of n x m real numbers for each $i = 1,2,\ldots,p$. The $i^{\underline{th}}$ isolated subsystem of interconnected

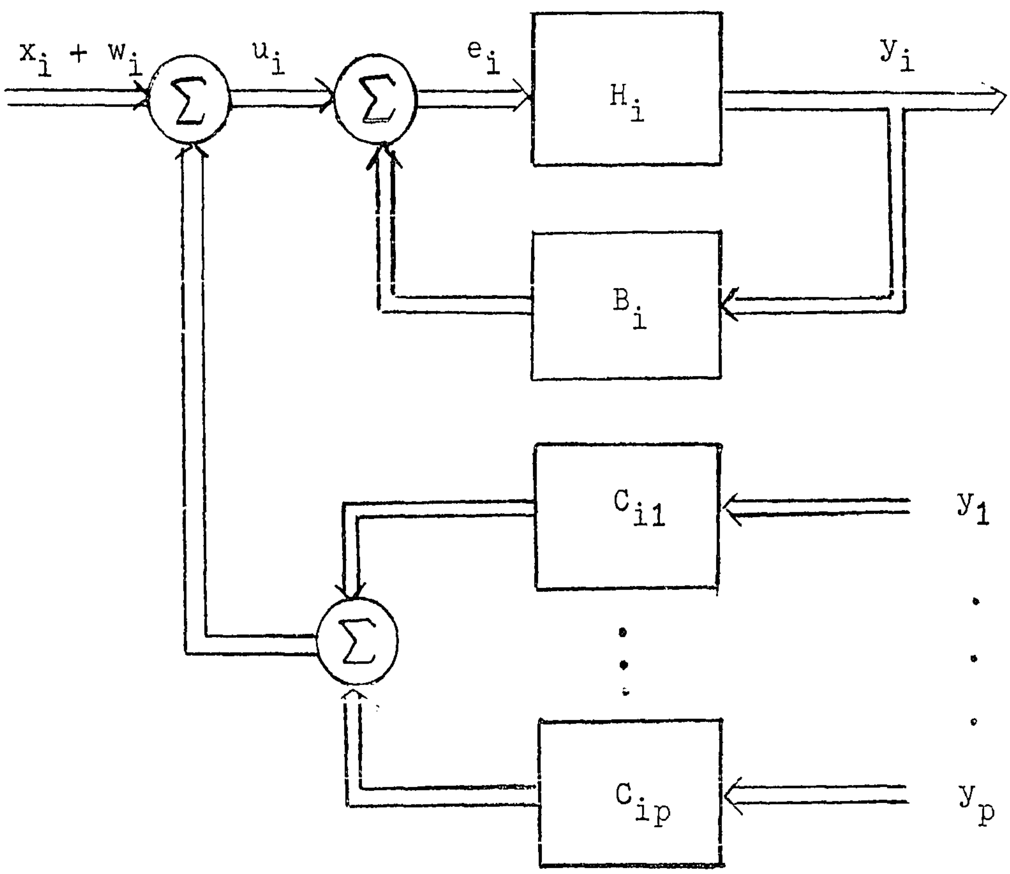


Figure 3: Interconnected feedback system (10).

system (10) is the system described by equations (10) (for that particular i) with C_{ij} set equal to zero for each $j = 1, 2, \ldots, p$. Thus, each isolated subsystem of interconnected system (10) has the form of the single-loop multiple-input multiple-output system considered in Chapter Three, in general, and Definitions 9 and 10 in particular.

Let U_{ij} and Y_{ij} represent the relations which map the input x_j into the error u_i or the output y_i, respectively. The stability of system (10) is interpreted as boundedness and continuity of these relations. Assuming that the $i^{\underline{th}}$ isolated subsystem of system (10) has margin of boundedness δ_i and gain factor μ_i for each $i = 1, 2, \ldots, p$, we have the following results, which are proved in Appendix C:

Theorem 5 All relations U_{ij} and Y_{ij} associated with interconnected system (10) are bounded if each of the gains $g(C_{ij})$ are finite and the successive principal minors of the test matrix $A = \left[a_{ij}\right]$ are all positive, where

$$a_{ij} = \begin{cases} \delta_j - g(C_{jj})\mu_j & \text{for } i = j \\ -g(C_{ij})\mu_j & \text{for } i \neq j \end{cases}$$

Remark 18 Positivity of the successive principal minors of the test matrix A can always be guaranteed, for fixed δ_j and $g(C_{ij})$, by making each gain factor sufficiently small.

Theorem 6 Assuming that the $i^{\underline{th}}$ isolated subsystem of system (10) has margin of continuity δ_{ci} and incremental gain factor μ_i^I for each $i = 1,2,\ldots,p$, all relations U_{ij} and Y_{ij} associated with interconnected system (10) are continuous if each of the incremental gains $g^I(C_{ij})$ are finite and the successive principal minors of the test matrix $A = [a_{ij}]$ are all positive, where

$$a_{ij} = \begin{cases} \delta_{cj} - g^I(C_{jj})\,\mu_j^I & \text{for } i = j \\ -g^I(C_{ij})\,\mu_j^I & \text{for } i \neq j \end{cases}$$

CHAPTER SIX: INTERCONNECTED FEEDBACK SYSTEMS -- GENERAL RESULTS FOR L_∞- AND l_∞-BOUNDEDNESS AND CONTINUITY

In this Chapter, we discuss interconnected feedback systems of the same form discussed in Chapter Five. In the present discussion, however, we are concerned with L_∞- and l_∞-boundedness and continuity rather than L_2- and l_2-boundedness and continuity.

Thus, the system model under consideration is the one governed by the functional equations (10) and depicted in Figure 3. The present discussion shall treat the case of continuous time systems. The treatment of discrete time systems is quite similar and only the final boundedness and continuity conditions will be given for that case. Thus, in system (10), each input x_i, error u_i or e_i, and output y_i is assumed to belong to the extended function space $L_{\infty e}$, while each reference signal is assumed to belong to L_∞. The relations H_i, B_i, and C_{ij} are all assumed to have ranges and domains which are subsets of $L_{\infty e}$. The comments about the matrix nature of these relations and the vector nature of the various signals made in Chapter Five apply here without change. The $i^{\underline{th}}$ isolated subsystem of interconnected system (10) is said to have Property A if it has margin of boundedness δ_i for some negative weights σ_{i1} and σ_{i2} and some conicity constants $r_i \geqslant 0$ and c_i (see Definition 12). In the remainder of this Chapter, we assume that each isolated subsystem of the interconnected system under question has Property A. Let $\mu_i = \max\{\sigma_{i1}, \sigma_{i2}\}$ and let μ be any constant with $0 > \mu \geqslant \max_{i=1,2,\ldots,p}\{\mu_i\}$. There are two possibilities for each isolated subsystem, corresponding to the identifications of B_i and H_i with B and H, respectively, in the definition of margin of boundedness, or the opposite identifications (of B_i and H_i with H and B, respectively). Recalling Remark 1 and employing the estimates provided by the calculations presented in Remark 17, we have

$$\|e_i;t,\mu\| \leqslant \frac{|c_i| + r_i}{\delta_i \quad r_i} \|u_i;t,\mu\|$$

for each $i = 1,2,\ldots,p$, for either of these two possibilities. Substituting for u_i from (10) and using the triangle inequality, we obtain

$$\begin{aligned} \|e_i;t,\mu\| &\leqslant \frac{|c_i| + r_i}{\delta_i \quad r_i}\Big\{\|x_i + w_i;t,\mu\| + \\ &\quad + \sum_{j=1}^{p} \|C_{ij}H_je_j;t,\mu\|\Big\} \\ &\leqslant \frac{|c_i| + r_i}{\delta_i \; r_i}\Big\{\|x_i + w_i;t,\mu\| + \sum_{j=1}^{p} g_{L_2}(e^{-\mu}C_{ij}H_je^{\mu})\|e_j;t,\mu\|\Big\} \end{aligned}$$

provided that the indicated L_2-gains are finite. In the last step, we have used the result, true for any e in $L_{\infty e}$, that

$$\|He;t,\mu\| = \|(He)_t \, e^{-\mu t}\|_{L_2} = \|(e^{-\mu} He^{\mu} y)_t\|_{L_2}$$

$$\leq g_{L_2}(e^{-\mu} He^{\mu}) \, \|y_t\|_{L_2} = g_{L_2}(e^{-\mu} He^{\mu}) \, \|e;t,\mu\|$$

where $y_t = e_t \, e^{-\mu} \in L_2$. Defining two column vectors

$$E_{t,\mu} = (\|e_1;t,\mu\| , \|e_2;t,\mu\| , \ldots, \|e_p;t,\mu\|)'$$

$$X_{t,\mu} = (\|x_1;t,\mu\| , \|x_2;t,\mu\| , \ldots, \|x_p;t,\mu\|)'$$

and a matrix $M = \left[m_{ij} \right]$ where

$$m_{ij} = \begin{cases} \dfrac{\delta_i \quad r_i}{|c_i| + r_i} - g_{L_2}(e^{-\mu} C_{ii} H_i e^{\mu}) & \text{for } i = j \\ - g_{L_2}(e^{-\mu} C_{ij} H_j e^{\mu}) & \text{for } i \neq j \end{cases} \tag{11}$$

the previous result may be summarized by the matrix inequality

$$M \, E_{t,\mu} \leq X_{t,\mu} + W_{t,\mu}$$

Provided that the successive principal minors of M are all positive, M^{-1} exists and all its components (call them D_{ij}) are nonnegative. (See Appendix A of Reference 34.) Therefore, for each i = 1,2,...,p, we have

$$\|e_i;t,\mu\| \leq \sum_{j=1}^{p} D_{ij} \left\{ \|x_j;t,\mu\| + \|w_j;t,\mu\| \right\}$$

Using the inequality $\|x;t,\mu\| \leq (-2\mu)^{\frac{1}{2}} e^{-\mu t} \|x\|_{L_\infty}$, which holds for all $x \in L_\infty$, we find that

$$e^{\mu t}\|e_i;t,\mu\| \leq \sum_{j=1}^{p} (-2\mu)^{\frac{1}{2}} D_{ij} \left\{ \|x_j\|_{L_\infty} + \|w_j\|_{L_\infty} \right\}$$

provided that each x_j is restricted to lie in a bounded subset of $L_{\infty e}$. Now, if each forward loop relation H_i has decaying L_1-memory, then, from Remark 7, there exist positive constants G_i such that

$$\|y_i\|_{L_\infty} \leq G_i \left\{ \sum_{j=1}^{p} (-2\mu)^{\frac{1}{2}} D_{ij} \left[\|x_j\|_{L_\infty} + \|w_j\|_{L_\infty} \right] \right\}$$

for each i = 1,2,...,p, which means that the relations which connect system inputs to each system output are L_∞-bounded. If, in addition, each feedback relation B_i has finite L_∞-gain, then it is easy to see that the relations which connect system inputs to each error signal are also L_∞-bounded. We have therefore established the following result:

<u>Theorem 7</u> Assuming that each isolated subsystem possesses Property A, then the relations which connect inputs of the continuous time interconnected system (10) to outputs are each <u>L_∞-bounded</u> provided that each H_i has decaying L_1-memory, each gain $g_{L_2}(e^{-\mu} C_{ij} H_j e^{\mu})$ is finite, and the successive principal minors of the test matrix M, defined in (11), are all positive. If, in addition, each relation B_i has finite L_∞-gain, then the relations which connect inputs to errors are each L_∞-bounded also.

<u>Remark 19</u> If C_{ij} and H_j belong to class $\mathcal{C}$, then we have

$$g_{L_2}(e^{-\mu} C_{ij} H_j e^{\mu}) = \max_{\omega \in R^+} \left| C_{ij}(\mu + j\omega) H_j(\mu + j\omega) \right|$$

where the maximum on the right is guaranteed to be finite. Furthermore, if C_{ij} and H_j belong to class $\mathcal{C}$ and <u>if</u>

$$g_{L_2}(C_{ij} H_j) = \max_{\omega \in R^+} \left| C_{ij}(j\omega) H_j(j\omega) \right|$$

is finite, then the analyticity of the Laplace transform guarantees that there exists a weight $\mu < 0$ (with $|\mu|$ sufficiently small) for which $g_{L_2}(e^{-\mu} C_{ij} H_j e^{\mu})$ is finite and arbitrarily close to $g_{L_2}(C_{ij} H_j)$. Therefore, in cases in which every one of the relations C_{ij} and H_j belongs to class $\mathcal{C}$, boundedness is guaranteed if the test matrix obtained by replacing all μ-shifted Fourier transforms with unshifted (i.e., $\mu = 0$) Fourier transforms satisfies the indicated positivity conditions. (The point is that continuity of the determinant implies that the positivity conditions will continue to be satisfied for some negative μ with $|\mu|$ sufficiently small.) This observation may be of great utility in practical situations (especially if only experimental frequency responses are avaliable) due to the difficulty in obtaining the μ-shifted Fourier transforms. Of course, the boundedness conditions obtained by considering nonzero weights μ may be less conservative.

The $i^{\underline{th}}$ isolated subsystem of interconnected system (10) is said to have Property B if it has margin of continuity δ_{ci} for some negative weights σ_{i1} and σ_{i2} and some conicity constants $r_i \geq 0$ and c_i. Defining μ as before, an argument closely parallel to the one just given establishes the following result:

Theorem 8 Assuming that each isolated subsystem possesses Property B, then the relations which connect inputs of the continuous time interconnected system (10) to outputs are each L_∞-continuous provided that each H_i has incrementally decaying L_1-memory, each incremental gain $g^I_{L_2}(e^{-\mu} C_{ij} H_j e^{\mu})$ is finite, and the successive principal minors of the test matrix $M = \left[m_{ij} \right]$ are all positive, where

$$m_{ij} = \begin{cases} \dfrac{\delta_i \quad r_i}{|c_i| + r_i} - g^I_{L_2}(e^{-\mu} C_{ii} H_i e^{\mu}) & \text{for } i = j \\ - g^I_{L_2}(e^{-\mu} C_{ij} H_j e^{\mu}) & \text{for } i \neq j \end{cases}$$

If, in addition, each relation H_i has finite incremental L_∞-gain, then the relations which connect inputs to errors are each L_∞-continuous also.

Remark 20 Remark 19 is also useful in obtaining continuity conditions since, when the operators in question belong to class $\hat{C}$, incremental L_∞-gains and L_∞-gains coincide.

Finally, we briefly consider interconnected discrete time systems. All that is needed to adopt the preceding discussion to this case is to replace the space $L_{\infty e}$ by the space $l_{\infty e}$. The only changes are to replace L_∞-gains and the requirement of a decaying L_1-memory by l_∞-gains and the requirement of a decaying l_1-memory (along with corresponding changes in incremental quantities when discussing continuity). Such a discussion leads to the following result:

Theorem 9 Assuming that each isolated subsystem possesses Property A, then the relations which connect inputs of the discrete time interconnected system (10) to outputs are each l_∞-bounded provided that each H_i has decaying l_1-memory, each gain $g_{l_1}(e^{-\mu} C_{ij} H_j e^{\mu})$ is finite, and the successive principal minors of the test matrix $N = [n_{ij}]$ are all positive, where

$$n_{ij} = \begin{cases} \dfrac{\delta_i \; r_i}{|c_i| + r_i} - g_{l_2}(e^{-\mu} C_{ii} H_i e^{\mu}) & \text{for } i = j \\ - g_{l_2}(e^{-\mu} C_{ij} H_j e^{\mu}) & \text{for } i \neq j \end{cases}$$

If, in addition, each relation B_i has finite l_∞-gain, then the relations which connect inputs to errors are each l_∞-continuous also.

Remark 21 If C_{ij} and H_j belong to class $\mathcal{C}$, then we can easily formulate a remark analogous to Remark 19 for the present case. The only significant difference is that in the discrete time case we deal with μ-shifted z-transforms rather than the μ-shifted Fourier transforms encountered in the continuous time case. A continuity result can also be stated for discrete time systems in close analogy to the result in Theorem 8.

CHAPTER SEVEN: POPOV-LIKE CONDITIONS FOR CONTINUOUS TIME INTERCONNECTED FEEDBACK SYSTEMS

The system model treated in this Chapter is a special case of the system governed by equations (10) and shown in Figure 3 for n = 1 and m = 1. Thus, we consider the stability problem for that class of systems which can be viewed as an interconnection of p scalar-input scalar-output subsystems. In addition, we assume that each isolated subsystem is of the Popov type. More specifically, the $i^{\underline{th}}$ isolated subsystem is assumed to consist of an operator $N_i \in \mathcal{N}$ in cascade with an operator $L_i \in \mathcal{L}$. We restrict ourselves to continuous time systems for which the underlying extended function space is L_{2e}. For the sake of simplicity, we further assume that each interconnecting relation C_{ij} belongs to class $\mathcal{L}$. We shall consider two different basic system configurations and various sets of assumptions on the system elements. In every case, however, we assume that each operator L_i admits a factorization of a particular type. Namely, we assume that we may write $L_i = L_{i2}L_{i1}$ where L_{i1} is a linear mapping of L_{2e} into itself, L_{i2} is a linear mapping of L_{2e} into a subset of itself, denoted by L_S, and that there exists a time-invariant linear mapping L_{i3} of L_S into L_{2e} such that $L_{i3}L_{i2} = I_{L_{2e}}$ (the identity operator on L_{2e}) and $L_{i2}L_{i3} = I_{L_S}$ (the identity operator on L_S). In what follows we choose $L_{i2} \in \mathcal{L}$ to be characterized by the Fourier transform $L_{i2}(j\omega) = (1+j\omega q_i)^{-1}$ for some constant $q_i > 0$. (For this choice of L_{i2}, L_S is simply the set of all functions in L_{2e} having time derivatives in L_{2e}.) The requirement that $|j\omega L_i(j\omega)|$ (as well as $|L_i(j\omega)|$ itself) be bounded for $\omega \in R^+$ is sufficient to ensure that $L_{i1} = L_{i3}L_i$ maps L_{2e} into itself and that $g_{L_2}(L_{i1})$ is finite.

First, consider the interconnected system shown in Figure 4a and governed by the following set of functional equations:

$$
\begin{aligned}
e_i &= x_i + w_i - L_i y_i + \sum_{j=1}^{p} C_{ij} y_j, \\
y_i &= N_i e_i,
\end{aligned}
\tag{12}
$$

for $i = 1,2,\ldots,p$. Here, each input x_i, error signal e_i, and output y_i is assumed to belong to L_{2e}, while each reference signal w_i is assumed to belong to L_2.

The $i^{\underline{th}}$ isolated subsystem of interconnected system (12) is the single-loop feedback system obtained by setting $C_{ij} = 0$ for $j = 1,2,\ldots,p$ (and that particular i) in (12). We make the following assumptions concerning the isolated subsystems: For each i, we assume that N_i is inside the sector $\{0, b_i\}$ for

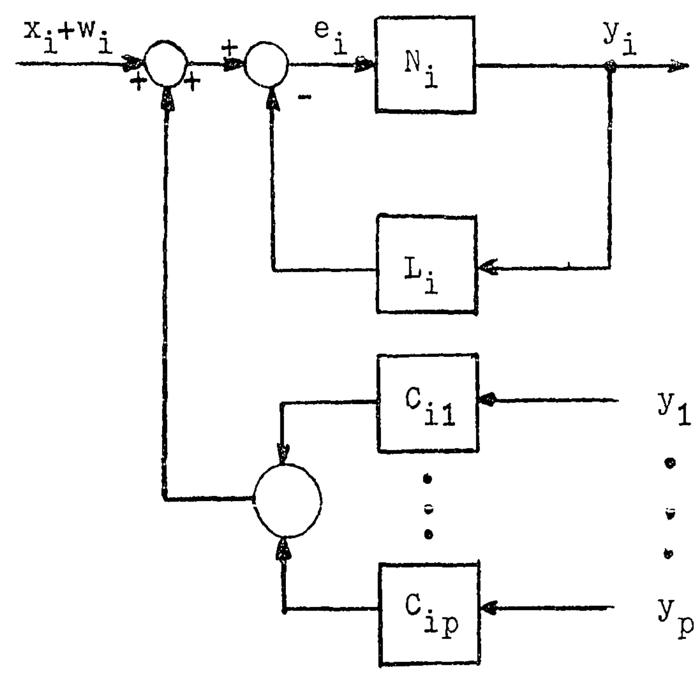


Figure 4a: Interconnected feedback system (12).

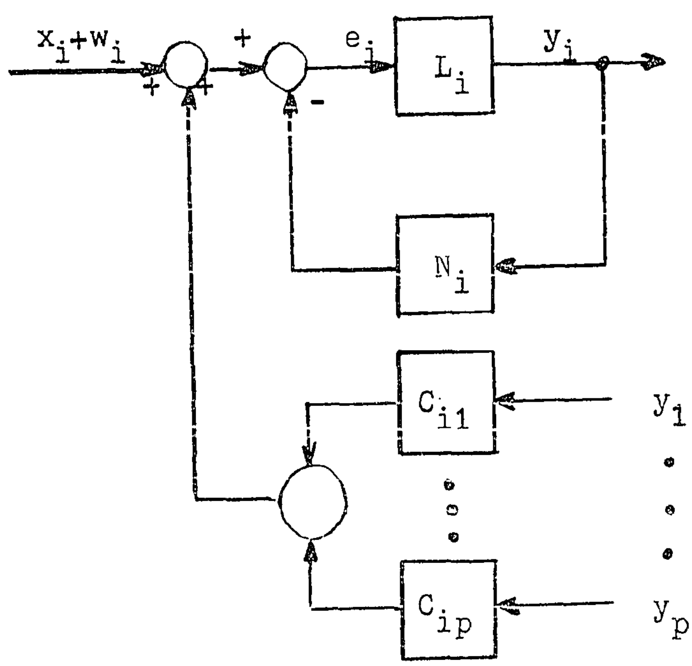


Figure 4b: Interconnected feedback system (14).

for some b_i with $0 < b_i < \infty$. We assume that each L_i has a factorization $L_i = L_{i2}L_{i1}$ of the type just discussed with the additional properties that:

(i) $|L_i(j\omega)|$ and $|j\omega L_i(j\omega)|$ are bounded for $\omega \in R^+$.

(ii) The operator $L_{i1} - \delta_i + b_i^{-1}$ is positive for some constant $\delta_i > 0$.

We then have the following result:

Theorem 10 Given the above assumptions concerning each isolated subsystem, interconnected system (12) is bounded if each of the gains $g_{L_2}(L_{i3}C_{ij})$ is finite and the successive principal minors of the test matrix $A = [a_{ij}]$ are all positive, where

$$a_{ij} = \begin{cases} \delta_j - g_{L_2}(L_{j3}C_{jj}) & \text{for } i = j \\ - g_{L_2}(L_{i3}C_{ij}) & \text{for } i \neq j \end{cases}$$

Remark 22 Boundedness of interconnected system (12) simply means that inputs x_i, w_i which belong to $L_S \cap L_2$ (i.e., which are such that x_i, $\dot{x}_i$, w_i, and $\dot{w}_i$ each belong to L_2) result in outputs $y_i \in L_2$.

Remark 23 Since $C_{ij} \in \mathcal{L}$ by assumption, we have

$$g_{L_2}(L_{i3}C_{ij}) = \operatorname*{ess\,sup}_{\omega \in R^+} |(1 + j\omega q_i)\, C_{ij}(j\omega)|$$

so that the gains which appear in the test matrix may each be calculated by a straightforward maximization of a function of a single real variable.

Remark 24 Condition (ii) on the operator L_{i1} is equivalent to the requirement that

$$\mathrm{Re}\ \left[(1+j\omega q_i)L_i(j\omega)\right] + b_i^{-1} \geq \delta_i \geq 0 \qquad (13)$$

for all $\omega \in R^+$, which is just the familiar Popov condition. If this condition is satisfied for a particular choice of q_i and for some value of δ_i, then that value of δ_i shall be termed a margin of boundedness of the i^{th} isolated subsystem. (The same term was employed earlier in this paper in a slightly different context and with a slightly different definition, but there seems little chance for confusion between the two usages. Each has the interpretation of a measure of the degree by which a certain stability criterion for that subsystem is satisfied. Note that, whereas the earlier margin of boundedness had to lie in the open interval (0,1), the present one can take on any positive value.) The present margin of boundedness has a simple graphical interpretation which is easily deduced from (13); namely, it is the minimum distance (parallel to the real axis) between the graph of the modified frequency response of the linear operator L_i and the Popov line with intercept $-b_i^{-1}$ and slope q_i^{-1}. Inspection of the test matrix of Theorem 10 reveals that interconnected system (12) is necessarily bounded (given our other assumptions) provided that the margin of boundedness of each isolated subsystem is sufficiently large. The proof of Theorem 10 (and of Theorems 11 and 12 which follow) is given in Appendix D.

Next, consider the similar system shown in Figure 4b and governed by the following set of functional equations:

$$\begin{aligned} e_i &= x_i + w_i - N_i y_i + \sum_{j=1}^{p} C_{ij} y_j, \\ y_i &= L_i e_i, \end{aligned} \tag{14}$$

for $i = 1,2,\ldots,p$. For this system, we make exactly the same assumptions as for interconnected system (12). In particular, those assumptions guarantee that $|L_{i1}(j\omega)|$ is bounded for $\omega \in R^+$, so that $g_{L_2}(L_{i1})$ is finite. Defining a parameter D_i for each $i = 1,2,\ldots,p$ by

$$D_i = \left\{ g_{L_2}(L_{i1}) \left[1 + g_{L_2}(L_{i1})/\delta_i \right] \right\}^{-1},$$

we have the following result:

Theorem 11 Given the above assumptions concerning each isolated subsystem, interconnected system (14) is bounded provided that all the gains $g_{L_2}(C_{ij}L_{j2})$ are finite and the successive principal minors of the test matrix $A = [a_{ij}]$ are all positive, where

$$a_{ij} = \begin{cases} D_j - g_{L_2}(C_{jj}L_{j2}) & \text{for } i = j \\ - g_{L_2}(C_{ij}L_{j2}) & \text{for } i \neq j \end{cases}$$

Remark 25 Boundedness of interconnected system (14) means that inputs $x_i, w_i \in L_2$ result in outputs $y_i \in L_2$. Note that

$$g_{L_2}(C_{ij}L_{j2}) = \operatorname*{ess\,sup}_{\omega \in R^+} |C_{ij}(j\omega)/(1+j\omega q_j)|$$

so that the factor L_{j2} serves to decrease the gain of the interconnection C_{ij} in general. Furthermore, note that as δ_i ranges from 0 to ∞, D_i ranges from 0 to $1/g(L_{i1})$ so that, in the present case, boundedness can not be guaranteed by making each δ_i sufficiently large.

Finally, we consider a system which is governed by the system of equations (14), but for which the operators N_i and L_i satisfy slightly different assumptions than before. Namely, we assume that for each i we have $N_i \in \mathcal{N}$ with N_i inside the sector $\{a_i, a_i+b_i\}$ for real constants a_i and b_i such that $-\infty < a_i < 0$ and $0 < a_i+b_i < \infty$. Similarly, we assume a factorization $L_i = L_{i2}L_{i1}$ of the usual type for each i, but now require that L_{i2} (still of the Popov type) be such that $-L_{i1}$ is inside the sector $\{a_i^{-1}, (a_i+b_i/(1-b_i\delta_i))^{-1}\}$ for some $\delta_i > 0$. Introducing the transformed operator

$$L'_{i1} = (L_{i1}^{-1} + a_i I)^{-1},$$

where I is the identity operator on L_{2e}, together with the corresponding parameter D'_i defined by

$$D'_i = \left\{g_{L_2}(L'_{i1})\left[1+g_{L_2}(L'_{i1})/\delta_i\right]\right\}^{-1},$$

we have the following result:

<u>Theorem 12</u> Given the above assumptions on each isolated subsystem, interconnected system (14) is <u>bounded</u> provided that all the gains $g_{L_2}(C_{ij}L_{j2})$ are finite and the successive principal minors of the test matrix $A = [a_{ij}]$ are all positive, where

$$a_{ij} = \begin{cases} D'_j - g_{L_2}(C_{jj}L_{j2}) \text{ for } i = j \\ -g_{L_2}(C_{ij}L_{j2}) \text{ for } i \neq j \end{cases}$$

Remark 26   The term boundedness has the same meaning here as in Remark 25.

Remark 27   The indicated transformation of L_{i1}, together with an appropriate transformation of the rest of the system (see Appendix D), results in a system of the type considered in Theorem 11. In particular, the operator $L'_{i1} - \delta_i + b_i^{-1}$ is guaranteed to be positive. Thus, the number δ_i can still be termed a margin of boundedness of the $i^{\underline{th}}$ isolated subsystem, although its graphical interpretation in terms of the Nyquist plot of L_{i1} (but not L'_{i1}) has necessarily been changed.

Remark 28   The present methods have not succeeded in obtaining corresponding continuity conditions for interconnected Popov systems. One of the difficulties besetting such a development is the fact that two signals whose difference is small (in the sense of having a small L_2 norm) do not necessarily have time derivatives whose difference is small.

CHAPTER EIGHT: ANALYSIS AND DESIGN OF INTERCONNECTED SYSTEMS

The results obtained in this paper, together with those of Reference 34, permit considerable flexibility of approach in the design and analysis of interconnected systems in the sense that a single system may be treated in several different ways. Among these various approaches, we believe that the one associated with Theorems 5 and 6 offers the most promise for the design and analysis of large-scale systems -- the advantage increasing with the dimension of the system. Thus, we contend that -- where possible due to the structure of the system and where desirable due to its complexity, the designer ought to view a multiple-input multiple-output system as the interconnection of single-loop feedback systems. In order to appreciate this viewpoint, consider the (admittedly somewhat extreme) case of a system of the form shown in Figure 3 in which every one of the relations H_i, B_i, and C_{ij} $(i,j = 1,2,\ldots,p)$ is a nontrivial one -- i.e., is not just a constant multiplier, for a total of $p(p+2)$ relations. For the sake of argument, let $p = 10$, so that $p(p+2) = 120$. Among the various boundedness conditions given here and in Reference 34, the one which yields the least conservative results is Theorem 3 of Reference 34. In order to apply this Theorem, we regard each of the $p(p+2)$ nontrivial relations as a forward loop relation so that the resulting interconnections are just constant multipliers (either 0 or 1). In so doing, we cast the system into the form shown in Figure 2 with $i = 1, 2, \ldots, 120$ and each $B_{ij} = 0$ or 1. Although this approach does, in general, lead to the least conservative available boundedness conditions, it does have several potential drawbacks: (1) In order to apply the Theorem, the entire 120-dimensional system would have to be transformed in a nontrivial way. (2) The resulting test matrix is 120 x 120 -- so the resulting positivity conditions are both numerous and complicated. (3) If the boundedness conditions are not met, this approach gives little guidance as to what modifications to make in order to enhance boundedness.

As an intermediate point of view -- one which considerably simplifies the boundedness conditions at the expense of obtaining conditions which are, in general, more conservative -- select a number of these $p(p+2) = 120$ relations to be considered as forward-loop relations and regard the rest as comprising the interconnecting structure. Thus, we force the system into the form shown in Figure 2 with nontrivial feedback relations B_{ij}. The number of relations which are regarded as forward-loop relations is somewhat arbitrary. For purposes of illustration, consider the $p = 10$ relations H_i of Figure 3 to be the forward-loop relations. We can then apply Theorem 3 directly (without transforming the system) and obtain a stability condition which requires all the successive principal minors of a 10 x 10 matrix to be positive. In certain special cases, the 120 positivity conditions found in the previous approach might be equivalent to these 10 conditions, but, in general, we will obtain 10 conditions which, taken together, are more stringent than the 120 ones found before. Although this smaller number of boundedness conditions certainly makes trial and error design techniques easier, we still are in the situation in which, if these conditions are not satisfied by a particular system, the present viewpoint does not suggest a systematic way of enhancing boundedness (other than to reduce the gains of all feedback relations).

Finally, consider the viewpoint adopted in Chapters Five, Six, and Seven of the present paper. Namely, choose to view the present multiple-input multiple-output system as consisting of p single-loop feedback systems interconnected by p x p other relations. (Because of the special form assumed for the present example, it is natural to let p = 10 here, but, in general, the designer is free to focus attention on as few (or as many) nonoverlapping feedback loops as he wishes with a corresponding gain (or loss) of conservatism in the boundedness conditions.) In order to apply Theorem 5 (similar steps are taken when applying Theorems 6, 7, 8, and 9 and, with a few modifications, Theorems 10, 11, and 12 as well), proceed as follows:

<u>Step 1</u> Impose the constraints that each single loop have margin of boundedness δ_i with $0 < \delta_i < 1$. Calculate the corresponding gain factor μ_i.

<u>Step 2</u> Form the p x p test matrix of Theorem 5. Boundedness conditions are obtained by requiring the successive principal minors of this matrix to be positive.

<u>Step 3</u> If the boundedness conditions are not all satisfied, modify some or all of the isolated subsystems in order to increase δ_i and decrease μ_i. Then repeat Step 2.

Compared to the second point of view suggested above, this approach results in boundedness conditions which, together with the auxiliary margin of boundedness conditions, are <u>more</u> conservative. This approach has the distinct and possibly decisive advantage, however, that it singles out a class of modifications which can be performed on each <u>isolated subsystem separately, but have the effect of enhancing overall boundedness.</u>

This advantage appears in a more intuitively appealing fashion if we shift our viewpoint from that of analysis to one of design. For example, given a multiple-input multiple-output system of the form depicted in Figure 2 which fails to meet the boundedness criterion of Theorem 3, how do we compensate this system in order to achieve boundedness? One way to tackle this exceedingly complex problem is to try local feedback. To this end, place a compensating feedback relation B_i around each "forward-loop" relation H_i -- each B_i being chosen so that the resulting single loop is bounded with margin of boundedness δ_i. The compensated system then has the form shown in Figure 3. Construct the test matrix of Theorem 5. If boundedness has not be ensured, alter each of the feedback relations (and the corresponding δ_i and μ_i) until it is.

The present approach is, by no means, a panacea for all design problems of multiple-input multiple-output systems. However, it does offer a flexible design strategy which can be applied in a straightforward manner. Furthermore, the resulting boundedness (or continuity) conditions reveal possible tradeoffs between the parameters describing the various isolated subsystems. By exploiting such tradeoffs (e.g., by decreasing the margin of boundedness in one subsystem while increasing it in others) the designer may be able to accommodate design specifications other than stability. Whenever a linear time-invariant (single-input single-output) element enters in an isolated subsystem, the margin of boundedness condition has a simple graphical interpretation in the Nyquist (or modified frequency response) plane. Thus, many of the frequency response compensation techniques developed for purely linear systems may be used to advantage here.

CHAPTER NINE: APPLICATIONS

In this Chapter, the previously derived results are used to study the stability of three specific interconnected continuous time systems. Although each example consists of more or less arbitrary interconnections of simple system elements (rather than concrete physical systems), the following treatments serve to illustrate the ease with which the present conditions may be applied and some of the flexibility they allow the system designer. As is the case with the study of absolute stability by the Second Method of Lyapunov, the present results yield stability conditions for entire classes of elements rather than particular choices of each element. In the first and third examples, the underlying extended function space is L_{2e}, while in the second it is $L_{\infty e}$. In every case we have $m = n = 1$ so that each system element is single-input single-output and each signal is scalar-valued.

Example 1

Consider the system shown in Figure 5a. This is a special case of the system of Figure 3 for $p = 3$ if we make the identifications

$$H_1 = G_1, \quad B_1 = -G_2, \; C_{21} = k_1, \; C_{31} = k_2,$$

$$H_2 = G_3, \quad B_2 = -G_5G_4, \quad C_{32} = 0, \quad C_{12} = G_9 - G_4, \tag{15}$$

$$H_3 = G_6, \quad B_3 = -G_7, \quad C_{13} = G_{10}, \quad C_{23} = G_8,$$

$$C_{11} = 0, \quad C_{22} = 0, \quad C_{33} = 0.$$

Here the operators G_1, G_3, G_4, and G_7 each belong to class $\mathcal{L}$ and are characterized by the indicated Laplace transforms; G_2, G_6, and G_8 belong to class $\mathcal{N}$ and are characterized by graphs which lie in the indicated shaded regions of the instantaneous input-output plane; G_5 and G_9 are time-varying linear gains; and G_{10} consists of an operator belonging to class $\mathcal{N}$ in cascade with a time-varying linear gain. The interconnections labelled by k_1 and k_2 are simple constant multipliers (k_1 and k_2 are just real constants).

For the sake of illustration, we shall regard the linear elements G_1, G_3, G_4, and G_7 as being adjustable. Only at the end shall we consider the particular Laplace transforms indicated in Figure 5a. In order to apply Theorem 5, we must first determine the margin of boundedness and gain factor of each isolated subsystem.

<u>Isolated subsystem 1</u> G_2 is interior conic $(3/4, 1/4)$. Setting $c_1 = -3/4$ and $(1-\delta_1)r_1 = 1/4$, we note that $c_1^2 > r_1^2$

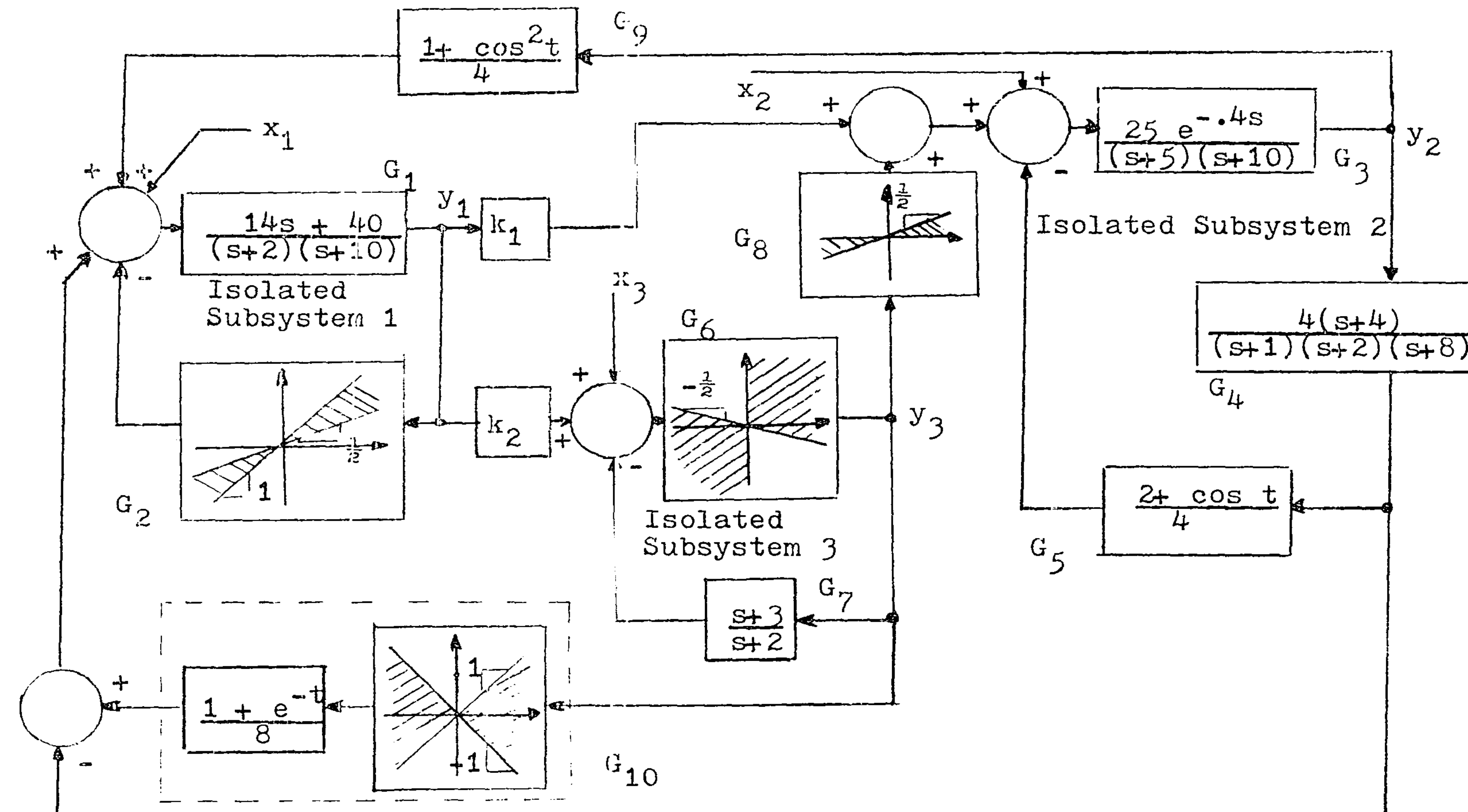


Figure 5a: Interconnected feedback system of Example 1.

provided that $2/3 > \delta_1 > 0$ and $r_1^2 > c_1^2$ if $1 > \delta_1 > 2/3$. Therefore, referring to Definition 9, isolated subsystem 1 will have margin of boundedness δ_1 with $2/3 > \delta_1 > 0$ if $-G_1$ lies outside the sector $\{4(1-\delta_1)/(4-3\delta_1), 4(1-\delta_1)/(2-3\delta_1)\}$ and $1 > \delta_1 > 2/3$ if $-G_1$ lies inside the same sector. From Definition 10, we see that, in either case, this isolated subsystem has gain factor μ_1 given by $\mu_1 = r_1^{-1} = 4(1-\delta_1)$.

<u>Isolated subsystem 2</u> G_5 has gain which varies with time in the range from .25 to .75 (where "gain" is meant in its generic sense -- the true operator gain of G_5 is, of course, independent of time and equal to .75). If we require G_3 and G_4 to be interior conic with constants $(0, b_3)$ and $(0, b_4)$, respectively, then this loop will have margin of boundedness δ_2 if the number $\delta_2 = 1 - .75 b_3 b_4$ lies in the open interval (0,1). (This can be verified by making the identifications $c_2 = 0$, $(1-\delta_2) r_2 = .75 b_4$, and $b_3 = r_2^{-1}$ in case (ii) of Definition 9.) From Definition 10, the corresponding gain factor is $\mu_2 = r_2^{-1} = b_3$.

<u>Isolated subsystem 3</u> G_6 is such that $2G_6 + I$ is positive. Identifying $r_3 = c_3 = 1$ in case (iii) of Definition 9, we see that this isolated subsystem will have margin of boundedness δ_3 if $-G_7$ is interior conic $(-1, (1-\delta_3))$. The corresponding gain factor is $\mu_3 = r_3^{-1} = 1$.

Once the above constraints have been imposed on each isolated subsystem, it only remains to form the test matrix of Theorem 5. Identifying the various gains from (15) and using the inequality $g_{L_2}(C_{12}) \leq .5+b_4$, we find that a suitable test matrix is

$$A = \begin{bmatrix} \delta_1 & -(.5+b_4)\mu_2 & -.25 \\ -|k_1|\mu_1 & \delta_2 & -.50 \\ -|k_2|\mu_1 & 0 & \delta_3 \end{bmatrix}$$

Requiring the successive principal minors of A to be positive yields a single independent condition:

$$\delta_1\delta_2\delta_3 - \left[(.5+b_4)\,|k_1|\mu_1\mu_2\delta_3 + |k_2|\mu_1\{.5(.5+b_4)\mu_2 + .25\delta_2\}\right] > 0$$

Using $\mu_1 = 4(1-\delta_1)$, $\delta_2 = 1-.75b_3b_4$, and $\mu_2 = b_3$, this may be re-expressed as

$$\delta_1\delta_2\delta_3 - (1-\delta_1)\left[(b_3+\tfrac{8}{3}(1-\delta_2))\,\{2|k_1|\,\delta_3 + |k_2|\} + |k_2|\delta_2\right] > 0 \qquad (16)$$

For each choice of δ_1, δ_2, δ_3, and b_3, we find a corresponding set (possibly empty) of values of k_1 and k_2 for which boundedness is guaranteed.

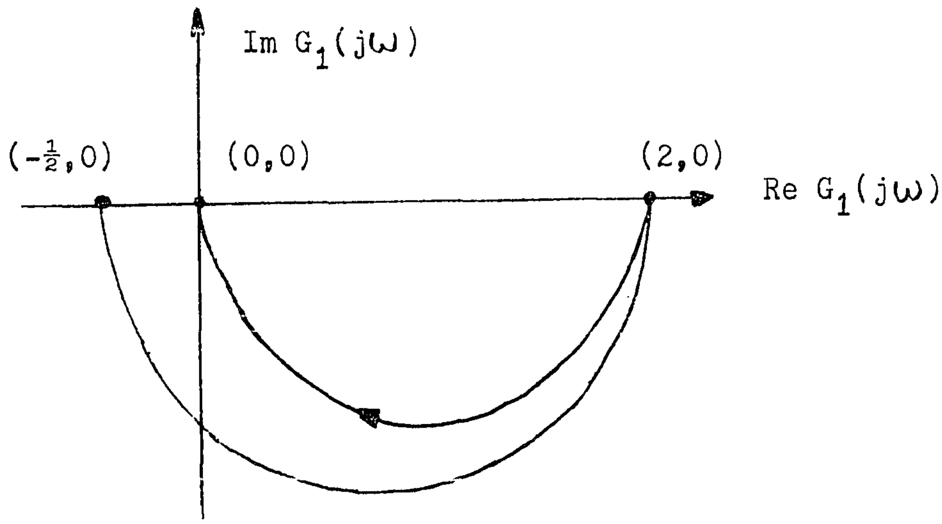


Figure 5b: Nyquist plot and associated conicity circle for linear element G_1 of Example 1.

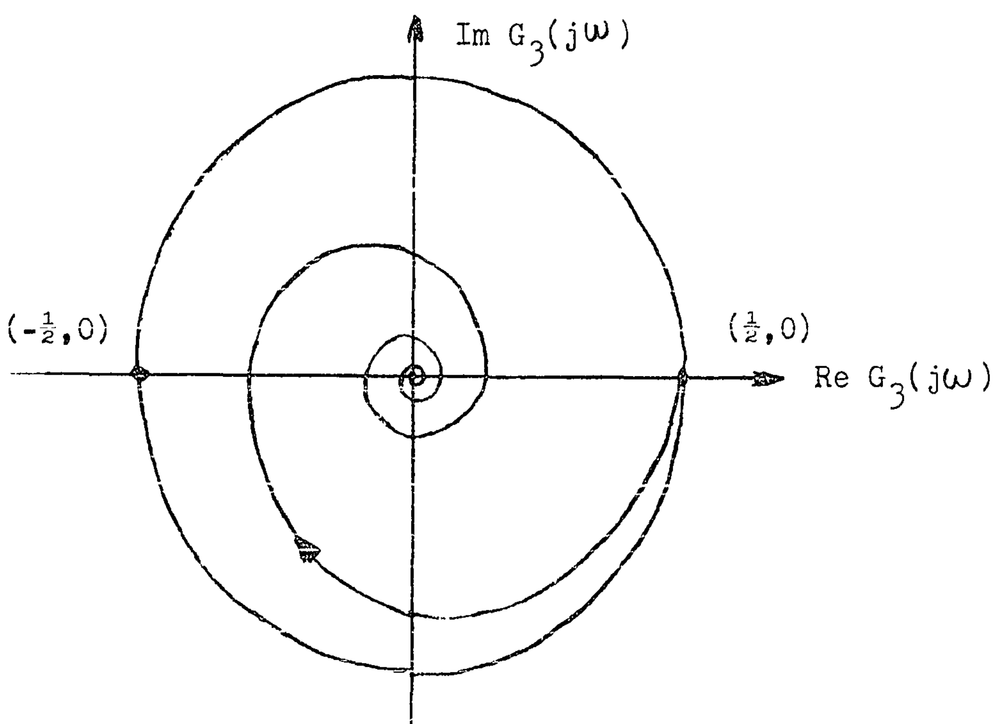


Figure 5c: Nyquist plot and associated conicity circle for linear element G_3 of Example 1.

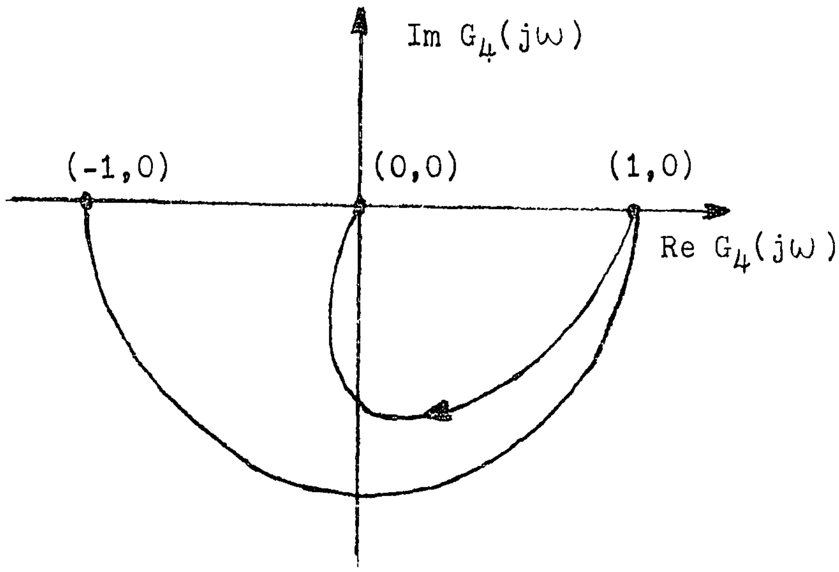


Figure 5d: Nyquist plot and associated conicity circle for linear element G_4 of Example 1.

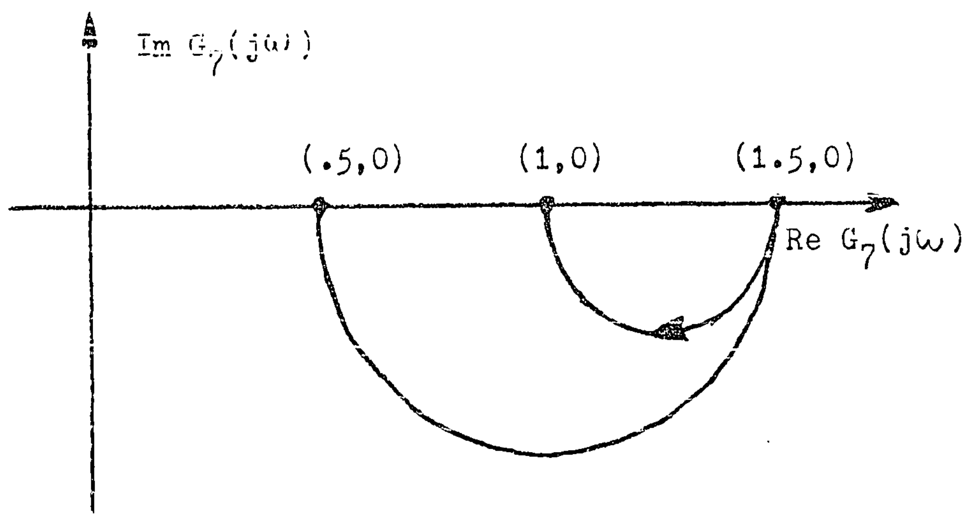


Figure 5e: Nyquist plot and associated conicity circle for linear element G_7 of Example 1.

We now consider the particular Laplace transforms indicated in Figure 5a. The Nyquist plot of G_1, shown in Figure 5b, shows that G_1 is inside the sector $\{-1/2, 2\}$. A brief calculation therefore shows that isolated subsystem 1 has margin of boundedness $\delta_1 = 4/5$. Similarly, $G_3(j\omega)$ and $G_4(j\omega)$ lie inside circles in the complex plane centered at the origin with radii 1/2 and 1, respectively (see Figures 5c and 5d). Therefore, $b_3 = 1/2$, $b_4 = 1$, and isolated subsystem 2 has margin of boundedness $\delta_2 = 5/8$. Finally, the Nyquist plot of G_7 shown in Figure 5e indicates that G_7 is inside the sector $\{1/2, 3/2\}$, which implies that $\delta_3 = 1/2$. Inserting these numbers into (16), we find that a sufficient condition for the boundedness of the system of Figure 5a (for <u>any</u> memoryless nonlinearities with graphs in the shaded regions) is given by

$$0 < 1.2|k_1| + 1.7|k_2| < 1.$$

Example 2

As an example of the application of Theorem 7, consider the simple continuous time interconnected system shown in Figure 6. The operators L_1, L_2, and L_3 are assumed to belong to class $\mathcal{C}$ and are characterized by the indicated Laplace transforms. The operators N_1, N_2, and N_3 are assumed to

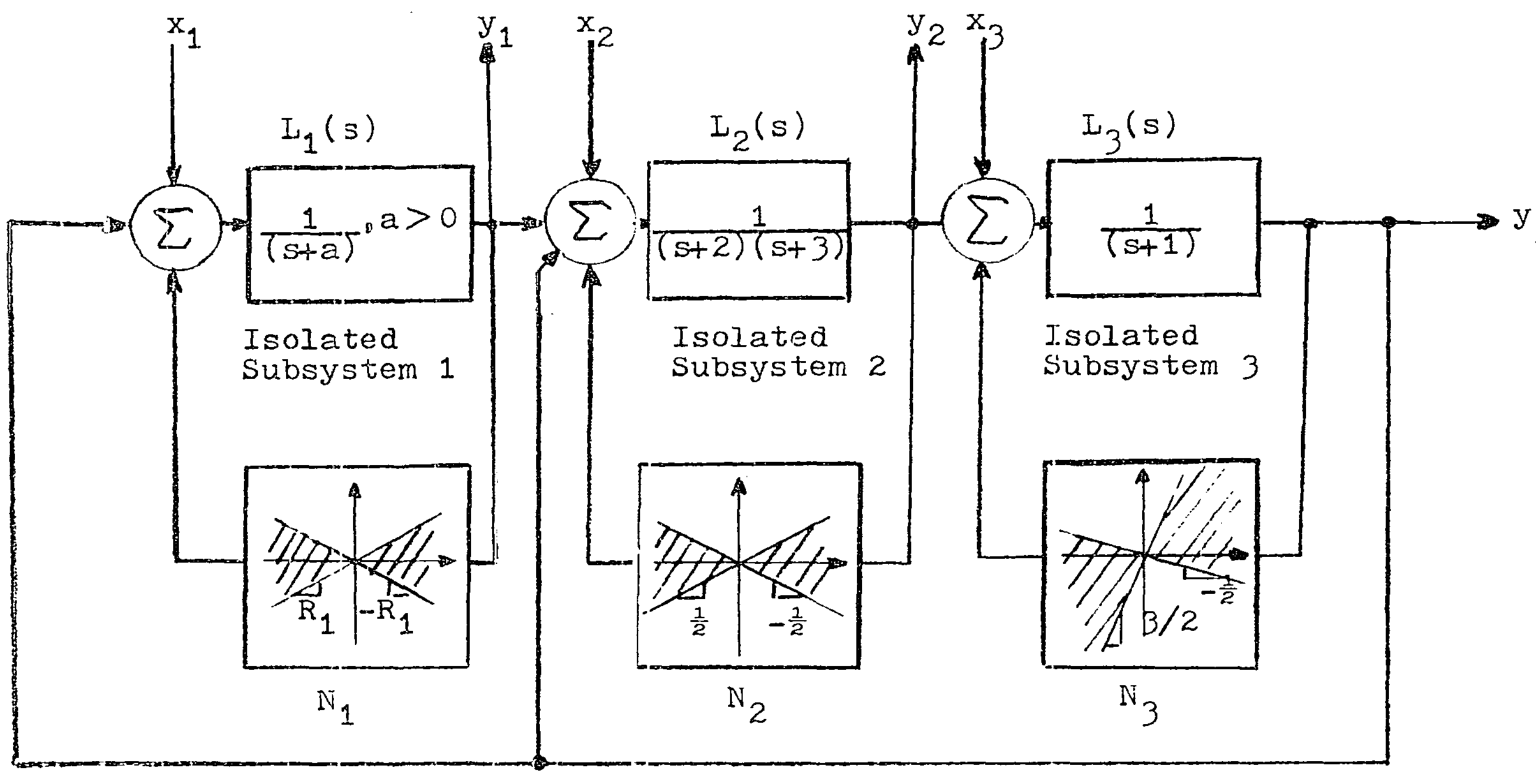


Figure 6: Interconnected feedback system of Example 2.

belong to class $\mathcal{N}$ and are characterized by graphs (not shown) in the input-output plane which are confined to the shaded regions indicated in the Figure. We consider this system to be the interconnection of the three isolated subsystems indicated in Figure 6. This system is, therefore, a special case of interconnected system (10) for $p = 3$ with $C_{21} = C_{13} = C_{23} = C_{32} = 1$ and all other C_{ij} equal to zero. (As mentioned before, the underlying extended function space is $L_{\infty e}$ in this example.)

In what follows, we take full advantage of Remark 19. Thus, we consider the unshifted Nyquist plots of the linear elements. When a conicity condition is referred to, no mention will be made of a particular weight. As indicated in Remark 19 (see also Remark 8), if the conditions derived later in this section are satisfied, then a negative weight μ (with $|\mu|$ sufficiently small) can be found so that L_∞-boundedness is guaranteed by Theorem 7 (all conicity, positivity conditions, etc., being interpreted in terms of that weight). More specifically, in what follows, we seek to establish a relationship between the two positive parameters a and R_1 in the first isolated subsystem sufficient to guarantee that the interconnected system be L_∞-bounded. The first step is to analyse each isolated subsystem:

Isolated subsystem 1 N_1 is interior conic $(0,R_1)$. Setting $c_1/(r_1^2 - c_1^2) = 0$ and $r_1/(r_1^2 - c_1^2) = R_1$, we find $c_1 = 0$ and $r_1 = 1/R_1$. This subsystem will have margin of boundedness δ_1 provided that L_1 is interior conic $(0, (1-\delta_1)r_1)$. This will be the case if $|(a+j\omega)^{-1}| \leq (1-\delta_1)r_1$ for all $\omega \in R^+$, which holds if and only if $1/a \leq (1-\delta_1)/R_1$. The best value of δ_1 is therefore given by

$$\delta_1 = 1 - R_1/a. \tag{17}$$

Isolated subsystem 2 N_2 is interior conic $(0,1/2)$. Setting $c_2/(r_2^2 - c_2^2) = 0$ and $r_2/(r_2^2 - c_2^2) = 1/2$, we find $c_2 = 0$ and $r_2 = 2$. Subsystem 2 will have margin of boundedness δ_2 provided that L_2 is interior conic $(0,(1-\delta_2)2)$. This will be the case if $|(2+j\omega)^{-1}(3+j\omega)^{-1}| \leq (1-\delta_2)2$ for all $\omega \in R^+$, which holds if and only if $1/6 \leq 2(1-\delta_2)$. The best value of δ_2 is therefore given by

$$\delta_2 = 1 - 1/12 = 11/12 \tag{18}$$

Isolated subsystem 3 N_3 is interior conic $(1/2,1)$. Setting $c_3/(r_3^2 - c_3^2) = 1/2$ and $r_3/(r_3^2 - c_3^2) = 1$, we find that $c_3 = 2/3$ and $r_3 = 4/3$. Subsystem 3 will have margin of boundedness δ_3 provided that L_3 is interior conic $(2/3, (1-\delta_3)4/3)$. This will be the case if $|(1+j\omega)^{-1} - 2/3|$

$\leq (1-\delta_3)4/3$ for all $\omega \in R^+$, which holds if and only if $2/3 \leq (1-\delta_3)4/3$. The best value of δ_3 is therefore given by

$$\delta_3 = 1 - 1/2 = 1/2. \qquad (19)$$

The next step is to form the test matrix of Theorem 7 for the interconnected system. Doing so, we have

$$M = \begin{bmatrix} \dfrac{\delta_1 r_1}{|c_1| + r_1} & 0 & -g_{L_2}(L_3) \\ -g_{L_2}(L_1) & \dfrac{\delta_2 r_2}{|c_2| + r_2} & -g_{L_2}(L_3) \\ 0 & -g_{L_2}(L_2) & \dfrac{\delta_3 r_3}{|c_3| + r_3} \end{bmatrix}$$

Using the previous calculations together with the gains $g_{L_2}(L_1) = 1/a$, $g_{L_2}(L_2) = 1/6$, and $g_{L_2}(L_3) = 1$, this becomes

$$M = \begin{bmatrix} \delta_1 & 0 & -1 \\ -1/a & 11/12 & -1 \\ 0 & -1/6 & 1/3 \end{bmatrix}$$

Requiring the successive principal minors of M to be positive results in three conditions, two of which are trivially satisfied. The remaining condition is

$$\delta_1 > 6/5a. \tag{20}$$

Combining (17) and (20), we obtain a single condition which is sufficient to ensure the L_∞-boundedness of the system of Figure 6, namely:

$$a > R_1 + 6/5$$

Thus, L_∞-boundedness is guaranteed if the single pole of L_1 is at least a distance $R_1 + 6/5$ into the left-half plane. As one might expect, the condition on L_1 becomes more restrictive (a must be larger) as the one on N_1 becomes less restrictive (R_1 larger) and vice versa.

It is worth noting that if the nonlinear elements N_1, N_2, and N_3 are <u>incrementally</u> inside the sectors indicated in Figure 6, then the condition derived above is sufficient to ensure that the system is L_∞-continuous.

Example 3

As an example of the application of Theorem 10, consider the system shown in Figure 7a. This system is a special case

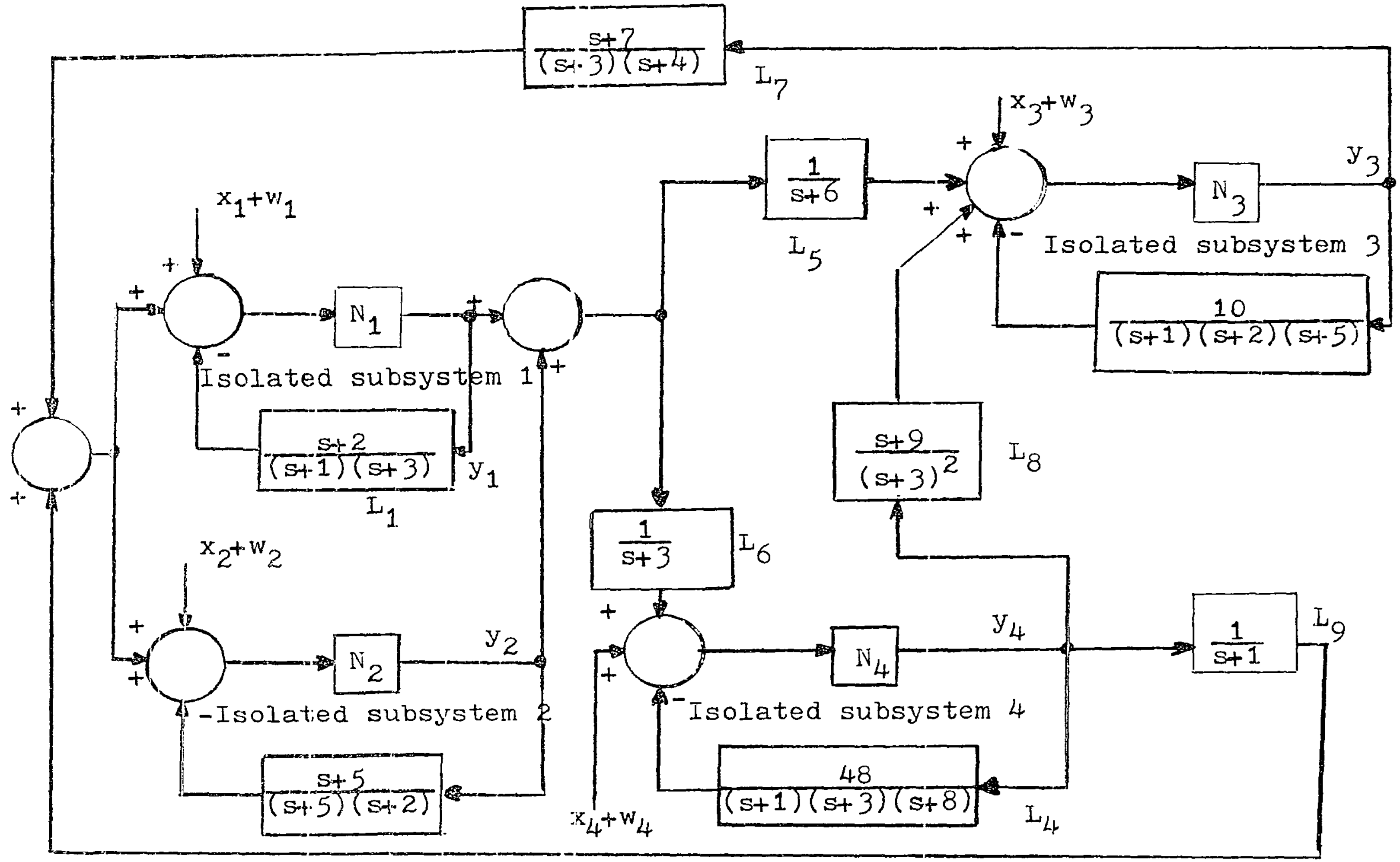


Figure 7a: Interconnected feedback system of Example 3.

of interconnected system (12) if we make the identifications

$$\begin{aligned}
&C_{11} = C_{22} = C_{33} = C_{44} = 0,\\
&C_{31} = C_{32} = L_5,\ C_{34} = L_8,\\
&C_{41} = C_{42} = L_6,\ C_{43} = 0, \qquad\qquad (21)\\
&C_{12} = C_{21} = 0,\ C_{13} = C_{23} = L_7,\\
&C_{14} = C_{24} = L_9.
\end{aligned}$$

In particular, each N_i is assumed to belong to class $\mathcal{N}$ and to satisfy a condition of the form $0 \leqslant N_i(x)/x \leqslant b_i < \infty$ for all real $x \neq 0$ and some positive constant b_i. Each L_i is assumed to belong to class $\mathcal{L}$ and is characterized by the indicated Laplace transform, which is denoted by $L_i(j\omega)$ for $i = 1,2,3,4$ and by the appropriate $C_{ij}(j\omega)$ (see equations (21)) for other values of i. Since the quantities $|L_i(j\omega)|$ and $|j\omega L_i(j\omega)|$ are bounded for $\omega \in R^+$ and each $i = 1,2,3,4$ (see the comments preceding the statement of Theorem 10), we are able to use Theorem 10 to study the boundedness of this system.

Here, we seek a set of restrictions on the nonlinearities which is sufficient to ensure boundedness. Thus, for each of the four isolated subsystems indicated in Figure 7a, we must find a Popov line with intercept $-b_i^{-1}$ and slope q_i^{-1} so that the plot of the modified frequency response of the corresponding linear element L_i lies entirely to the right of this line, the distance between the two (measured parallel to the real axis) being the margin of boundedness of that subsystem. Plots of the modified frequency responses of L_1, L_2, L_3, and L_4 are given in Figures 7b-7e (together with the particular Popov lines to be discussed below).

Assume, for the moment, that a suitable set of Popov lines has been found and the corresponding margins of boundedness identified. The next step is to form the test matrix of Theorem 10. Boundedness is then ensured if the successive principal minors of this test matrix are all positive. However, the choice of the Popov slope parameters q_i affects both the margins of boundedness and the various gains which enter in this test matrix in somewhat subtle ways. As a consequence, although these adjustable Popov slope parameters provide a desirable degree of flexibility to the present method, there does not seem to be a straightforward way of selecting them in an "optimal" manner.

Returning to the present example, we must calculate the

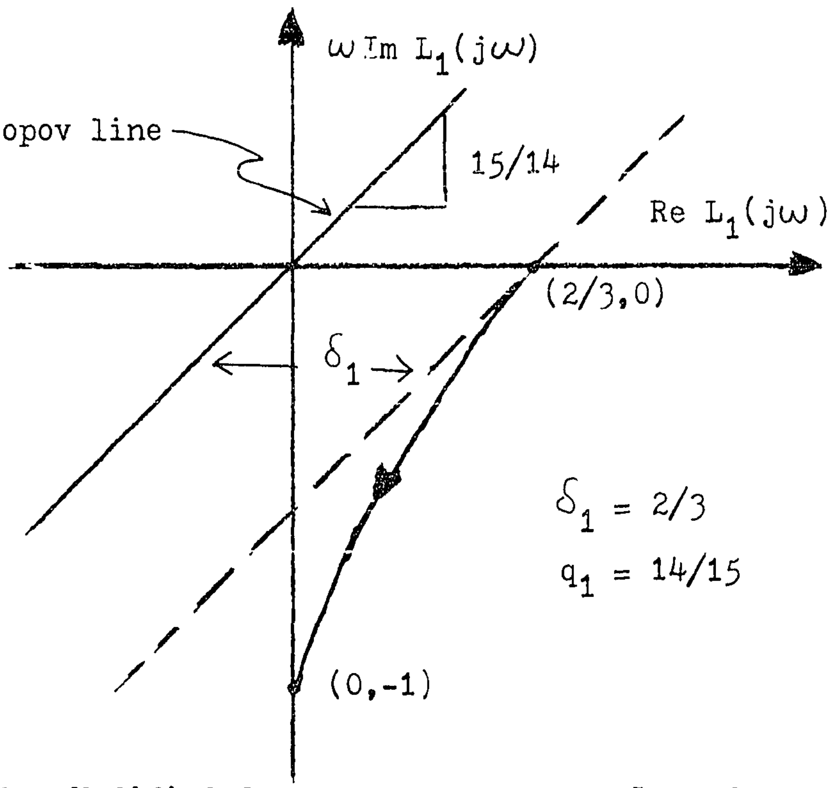


Figure 7b: Modified frequency response of L_1 and associated Popov line.

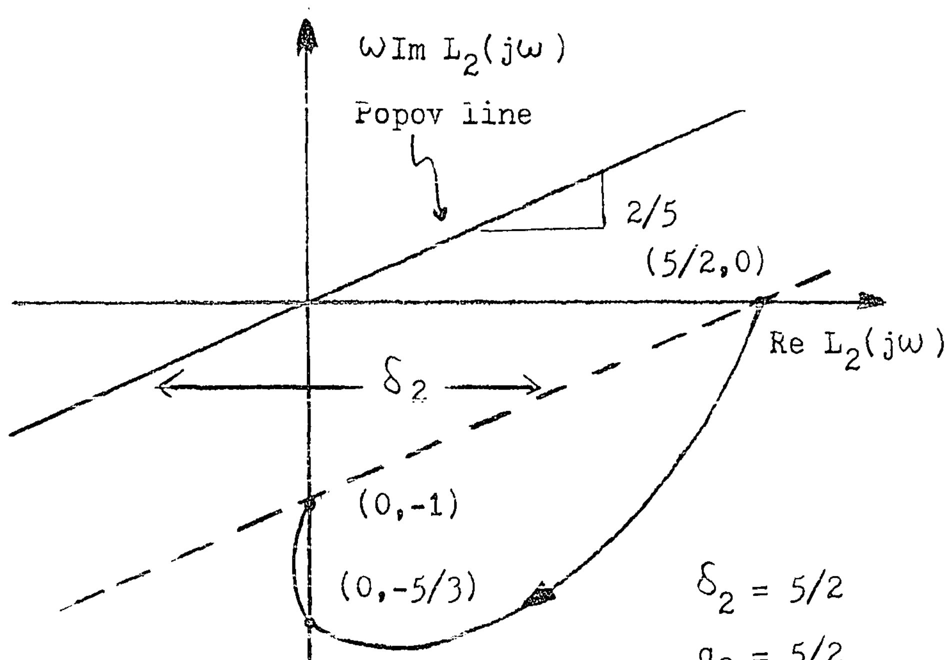
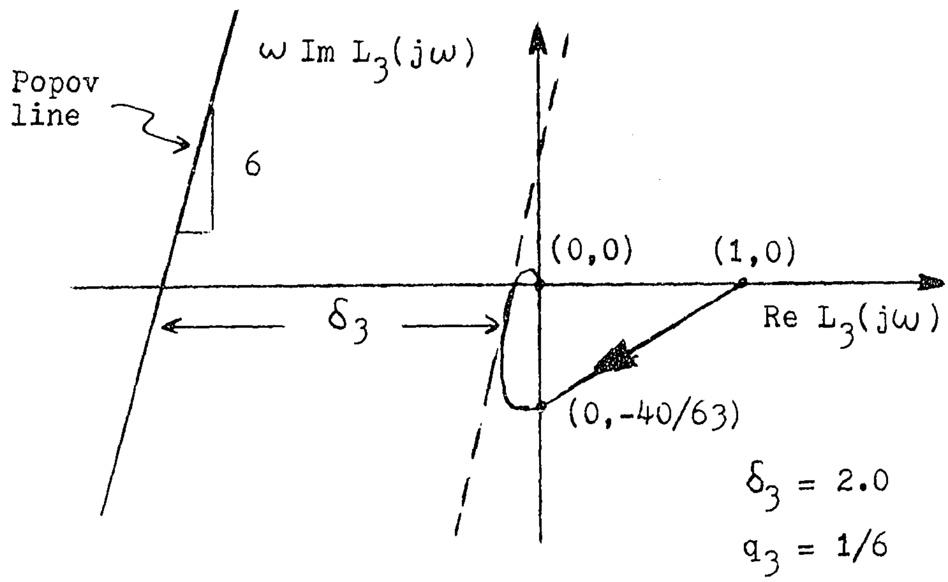


Figure 7c: Modified frequency response of L_2 and associated Popov line.



Figue 7d: Modified frequency response of L_3 and associated Popov line.

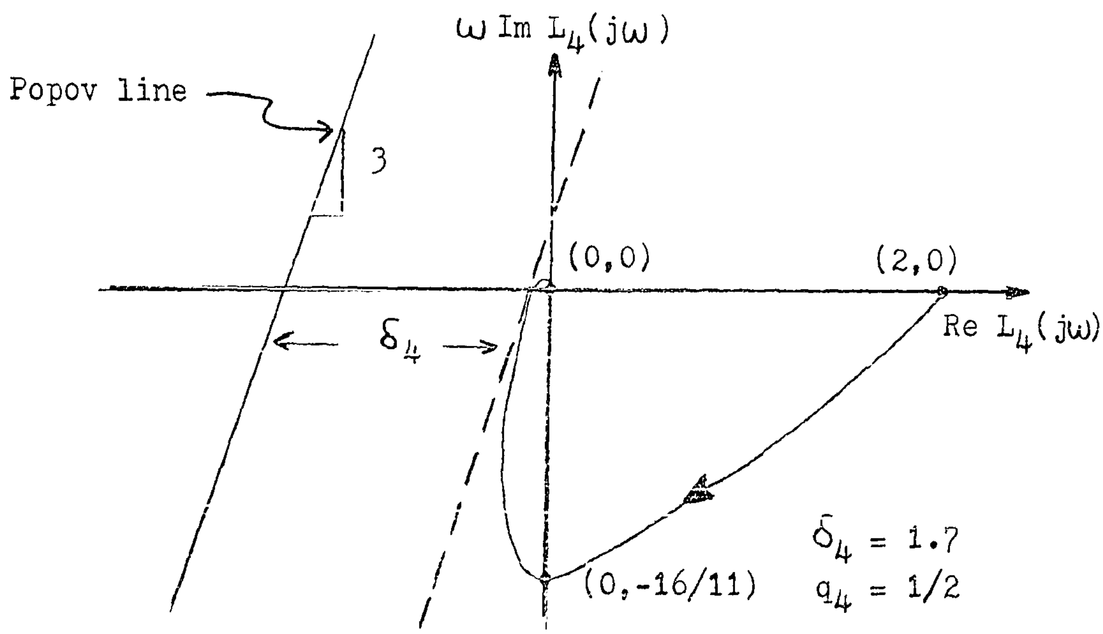


Figure 7e: Modified frequency response of L_4 and associated Popov line

various gains $g_{L_2}(L_{i3}C_{ij})$ which occur in the test matrix.
A sample calculation is

$$g_{L_2}(L_{33}C_{31}) = \max_{\omega \in R^+} \left|(1+j\omega q_3)(6+j\omega)^{-1}\right|$$

$$= \max(1/6, q_3)$$

In this manner, we may write the test matrix as follows (the subscript L_2 has been dropped from the remaining gains):

$$A = \begin{bmatrix} \delta_1 & 0 & -g(L_{13}C_{13}) & -\max(1,q_1) \\ 0 & \delta_2 & -g(L_{23}C_{23}) & -\max(1,q_2) \\ -\max(1/6,q_3) & -\max(1/6,q_3) & \delta_3 & -g(L_{33}C_{34}) \\ -\max(1/3,q_4) & -\max(1/3,q_4) & 0 & \delta_4 \end{bmatrix}$$

The remaining gains in A can not be specified in the simple closed form of the other entries until more is known about the parameters q_1, q_2, and q_3. Although the interplay between the various parameters is somewhat intricate, roughly speaking we should try to choose the q_i's to make the diagonal terms of A large and the off-diagonal terms small (in magnitude).

As an illustration of the type of boundedness information which can be obtained, suppose that we allow b_1 and b_2 to be arbitrarily large -- i.e., suppose we seek to derive conditions sufficient to ensure boundedness for <u>any</u> finite (but fixed) values of b_1 and b_2. In an attempt to make δ_1 and δ_2 as large as possible without making the off-diagonal elements of A unnecessarily large, choose $q_1 = 14/15$ and $q_2 = 5/2$. Then, as indicated in Figures 7b and 7c, we have $\delta_1 \geqslant 2/3$ and $\delta_2 \geqslant 5/2$ for any finite b_1 and b_2. For this q_1, we have $g_{L_2}(L_{13}C_{13}) = Q = \max \left|(1+j\omega q_1)(7+j\omega)(3+j\omega)^{-1}(4+j\omega)^{-1}\right|$. The maximum is achieved at $\omega^2 \cong 5.50$, which gives $Q \cong 1.01$. Similarly, for this q_2, we find that $g_{L_2}(L_{23}C_{23}) = P = \max \left|(1+j\omega q_2)(7+j\omega)(3+j\omega)^{-1}(4+j\omega)^{-1}\right| \cong 2.6525$, the maximum occurring at $\omega^2 \cong 7.695$.

In order to minimize the remaining off-diagonal terms, we also require $q_3 \leqslant 1/6$ and $q_4 \leqslant 1/3$. It is then easy to show that $g_{L_2}(L_{13}C_{13}) = \max \left|(1+j\omega q_3)(9+j\omega)(3+j\omega)^{-2}\right| = 1$, the maximum occurring at $\omega^2 = 0$. All the off-diagonal elements of A are now completely determined and we have

$$A = \begin{bmatrix} \delta_1 & 0 & -Q & -1 \\ 0 & \delta_2 & -P & -5/2 \\ -1/6 & -1/6 & \delta_3 & -1 \\ -1/3 & -1/3 & 0 & \delta_4 \end{bmatrix}$$

Requiring the successive principal minors of A to be positive and doing some algebra, we obtain the single independent condition

$$3\,\delta_1\,\delta_2\,\delta_3\,\delta_4 - (P\delta_1 + Q\delta_2)(1+\tfrac{1}{2}\delta_4) - (\tfrac{5}{2}\delta_1 + \delta_2)\,\delta_3 > 0$$

Setting δ_1 and δ_2 equal to their minimum possible values and substituting for P and Q, we find the condition

$$\delta_3\,\delta_4 - .833\;\delta_3 - .429\;\delta_4 > .854 \qquad (22)$$

Therefore, the interconnected system of Figure 7a will be bounded if we can find Popov lines for isolated subsystems 3 and 4 with $q_3 \leq 1/6$ and $q_4 \leq 1/3$ and with margins of boundedness satisfying (22). One suitable set of choices is $q_3 = 1/6$, $q_4 = 1/3$, $\delta_3 = 2.0$, and $\delta_4 = 1.7$, which leads to the Popov lines shown in Figures 7d and 7e. These choices are made most conveniently by graphical rather than analytic techniques. An approximate graphical analysis gives the bounds on the corresponding nonlinearities as $b_3 \cong .47$ and $b_4 \cong .55$.

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APPENDIX A

Interpretation of Gain Factor

Suppose that the system shown in Figure 1 and governed by equations (3) has margin of boundedness δ. For the sake of brevity, in what follows we shall replace the symbols $x + w$ and $u + v$ by x and u, respectively. Define two transformed relations H' and B' by $H' = (H^{-1} + cI)^{-1}$ and $B' = B + cI$. It is easy to show (cf., Reference 34) that system (3) is equivalent to the transformed system shown in Figure 8 and governed by the functional equations

$$
\begin{aligned}
y_i' &= \sum_{j=1}^{n} H_{ij}'e_j', \quad e_i' = x_i' + z_i', \\
z_i' &= \sum_{j=1}^{n} B_{ij}'f_j', \quad f_i' = u_i' + y_i',
\end{aligned}
\tag{3'}
$$

for $i = 1,2,\ldots,n$, where

$$x_i' = x_i - cu_i, \quad u_i' = u_i,$$

$$e_i' = e_i + cy_i, \quad f_i' = f_i,$$

$$y_i' = y_i, \quad z_i' = z_i + cf_i.$$

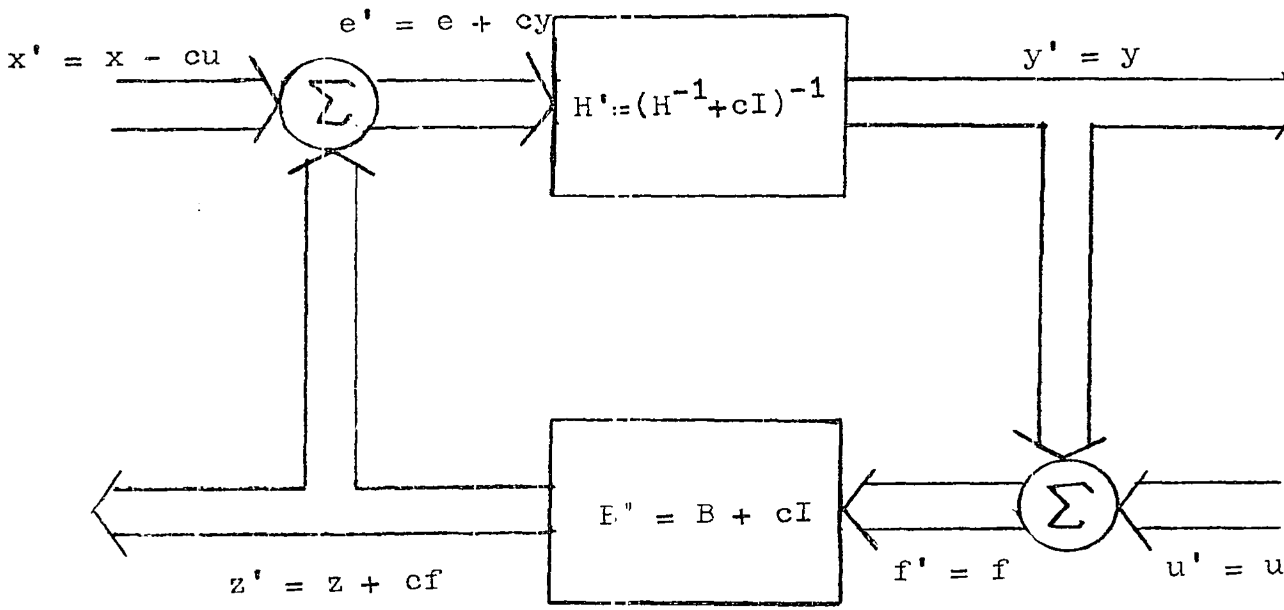

Figure 8: Transformed system equivalent to system (3) shown in Figure 1.

By assumption (see Definition 9), B is interior conic $(-c,(1-\delta)r)$, which implies that $B' = B + cI$ is interior conic $(0,(1-\delta)r)$. Using standard manipulations on conicity and positivity conditions, it is not difficult to show that $H' = (H^{-1} + cI)^{-1}$ is interior conic $(0,r^{-1})$ for all three conditions on H given in Definition 9.

In the special case in which $u = 0$, we have

$$||e_T'|| = ||x_T + (B'H'e')_T||$$

$$\leq ||x_T|| + g(B')g(H')\,||e_T'||$$

$$\leq ||x_T|| + (1-\delta)\,||e_T'||$$

for each $T \in S$ since $g(B') \leq (1-\delta)r$ and $g(H') \leq r^{-1}$. This implies that $\delta\,||e_T'|| \leq ||x_T||$. Since we also have

$$||y_T|| = ||(H'e')_T|| \leq g(H')\,||e_T'|| \leq r^{-1}\,||e_T'||,$$

we have $||y_T|| \leq (\delta r)^{-1}\,||x_T||$ for all $T \in S$ in this special case.

In the special case in which $x = 0$, we have

$$||f_T|| = ||u_T + (H'(-cu + B'f))_T||$$

$$\leq ||u_T|| + g(H') \left[|c|\,||u_T|| + g(B')\,||f_T||\right]$$

$$\leq ||u_T|| + r^{-1} \left[|c|\,||u_T|| + (1-\delta)r\,||f_T||\right]$$

This implies that

$$\delta\,||f_T|| \leq \left[1 + |c|\,r^{-1}\right] ||u_T||$$

Since

$$||z_T|| = ||-cf_T + (B'f)_T|| \leq \left[|c| + g(B')\right] ||f_T||$$

$$\leq \left[|c| + r^{-1}\right] ||f_T|| ,$$

we finally obtain $||z_T|| \leq \delta^{-1} (|c| + r^{-1})(1 + |c|\,r^{-1})\,||u_T||$

Together, these two calculations verify the statement made on p. 47 about the interpretation of the gain factor.

Proof of Equation (6)

Our goal in this section is to verify that Equation (6) holds for all three cases of Definition 12. First, consider case (i). By assumption, we have the inequalities

$$\left|\left|He + c(c^2-r^2)^{-1}e;t,\mu\right|\right| \geqslant r(r^2-c^2)^{-1} \left|\left|e;t,\mu\right|\right| \quad \text{(A1)}$$

$$\left|\left|Bf + cf;t,\mu\right|\right| \leqslant (1-\delta)r \left|\left|f;t,\mu\right|\right| \quad \text{(A2)}$$

But the former may be re-expressed as

$$\langle He,He;t,\mu\rangle + 2c(c^2-r^2)^{-1}\langle e,He;t,\mu\rangle +$$

$$c^2(c^2-r^2)^{-2}\langle e,e;t,\mu\rangle \geqslant r^2(r^2-c^2)^{-2}\langle e,e;t,\mu\rangle$$

Multiplying through by the negative number r^2-c^2, this becomes

$$(r^2-c^2)\langle He,He;t,\mu\rangle - 2c\langle e,He;t,\mu\rangle - \langle e,e;t,\mu\rangle \leqslant 0 \quad \text{(A3)}$$

Rearranging, this is equivalent to

$$r^2\langle He,He;t,\mu\rangle \leqslant \langle cHe + e,cHe + e;t,\mu\rangle$$

Expressing this in terms of the corresponding norms and taking the positive square root of each side gives (6). The corresponding algebra for case (ii) involves only a few sign changes and will not be given. Similarly, the condition which replaces (A1) for case (iii) is easily seen to give (A3) for the special case $r^2 = c^2$, which completes the demonstration.

APPENDIX B

Proofs of Theorems 3 and 4

From equations (9), we find

$$e_i = x_i + w_i + \sum_{j=1}^{p} B_{ij} H_j e_j. \qquad \text{(B1)}$$

Truncating this equation at time T, taking the norm of each side, using the triangle inequality and the definition of gain, this implies

$$\|e_{iT}\| \leq \|x_{iT}\| + \|w_{iT}\| + \sum_{j=1}^{p} g(B_{ij}H_j)\,\|e_{jT}\|$$

Introducing column vectors E_T, X_T, and W_T as in (1), this may be expressed as the matrix equation $T\,E_T \dot{\leq} X_T + W_T$ where T is the test matrix defined in Theorem 3. Under the positivity conditions given in the statement of the Theorem, T^{-1} exists and all its components are nonnegative. The rest of the proof of Theorem 3 then follows in the same manner as the proof of Theorem 1 given in Chapter Three. In order to prove Theorem 4, we consider (B1) for two different choices of the inputs x_i. Subtracting the resulting equations from each other and performing the same steps as in the previous proof upon the resulting equation establishes Theorem 4. (The only change is that gains are replaced by incremental gains.)

APPENDIX C

Proofs of Theorems 5 and 6

Consider the interconnected system shown in Figure 3 and governed by the system of equations (10). Assuming that the i^{th} isolated subsystem of this interconnected system has margin of boundedness δ_i and gain factor μ_i for each $i = 1,2,\ldots,p$, we have the estimates

$$||y_{iT}|| \leq (\mu_i/\delta_i)\,||u_{iT}||$$

for each $T \in S$. Therefore, using (10), the triangle inequality, the definition of gain, and the above estimates, we have

$$\begin{aligned} ||u_{iT}|| &\leq ||x_{iT}|| + ||w_{iT}|| + \sum_{j=1}^{p} ||(C_{ij}y_j)_T|| \\ &\leq ||x_{iT}|| + ||w_{iT}|| + \sum_{j=1}^{p} g(C_{ij})\,(\mu_j/\delta_j)\,||u_{jT}|| \end{aligned}$$

Introducing the column vectors U_T, X_T, and W_T as in (1), this may be written as a matrix inequality: $M\,U_T \leq X_T + W_T$ where $M = \left[m_{ij}\right]$ and

$$m_{ij} = \begin{cases} 1 - g(C_{jj})\,(\mu_j/\delta_j) & \text{for } i = j \\ -g(C_{ij})\,(\mu_j/\delta_j) & \text{for } i \neq j \end{cases}$$

If all the successive principal minors of the test matrix M are positive, then M^{-1} exists and all its components are non-negative. The proof of the boundedness of the relations U_{ij} and Y_{ij} is then easily completed, if this condition is satisfied, by following the final steps of the proof of Theorem 1 given in Chapter Three. Multiplying every element of the j^{th} column of M by the positive number δ_j for each $j = 1,2, \ldots,p$ results in the test matrix A given in the statement of Theorem 5. Since none of these multiplications can change the sign of any of the successive principal minors of M, A is an equivalent test matrix and Theorem 5 has been established. The proof of the corresponding continuity result (Theorem 6) follows in the usual fashion.

APPENDIX D

Proofs of Theorems 10, 11, and 12

Under the assumptions of Theorem 10, the interconnected system shown in Figure 4a and governed by equations (12) is equivalent to the transformed system shown in Figure 9a and governed by the following functional equations

$$
\begin{aligned}
v_i &= L_{i3}(x_i + w_i) - L_{i1}y_i + \sum_{j=1}^{p} L_{i3}C_{ij}y_j, \\
y_i &= N_i e_i, \quad e_i = L_{i2}v_i,
\end{aligned}
\tag{12'}
$$

for $i = 1,2,\ldots,p$. Truncating the first of these equations at time T and taking the inner product of the result with y_{iT}, we obtain

$$
\begin{aligned}
&\langle (L_{i3}(x_i + w_i))_T, y_{iT}\rangle + \sum_{j=1}^{p} \langle (L_{i3}C_{ij}y_j)_T, y_{iT}\rangle \\
&= \langle v_{iT}, y_{iT}\rangle + \langle (L_{i1}y_i)_T, y_{iT}\rangle \qquad \text{(D1)} \\
&= \langle v_{iT}, (N_iL_{i2}v_i)_T\rangle + \langle (L_{i1}y_i)_T, y_{iT}\rangle
\end{aligned}
$$

Now, by assumption, each $N_i \in \mathfrak{N}$ and is inside the sector $\{0, b_i\}$ for some $b_i > 0$. Therefore, since the multiplier L_{i2}

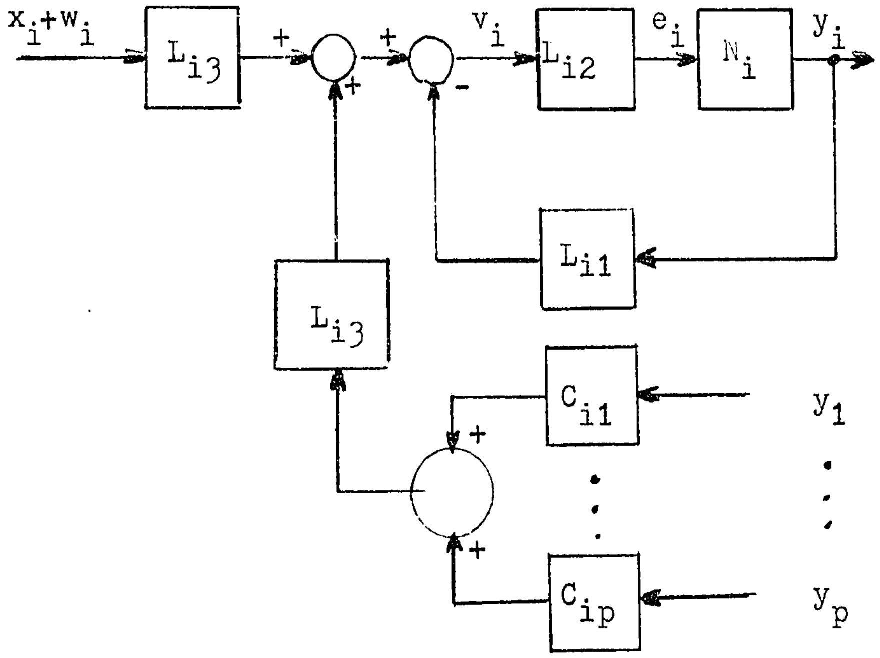

Figure 9a: Transformed system equivalent to interconnected system (12) of Figure 4a.

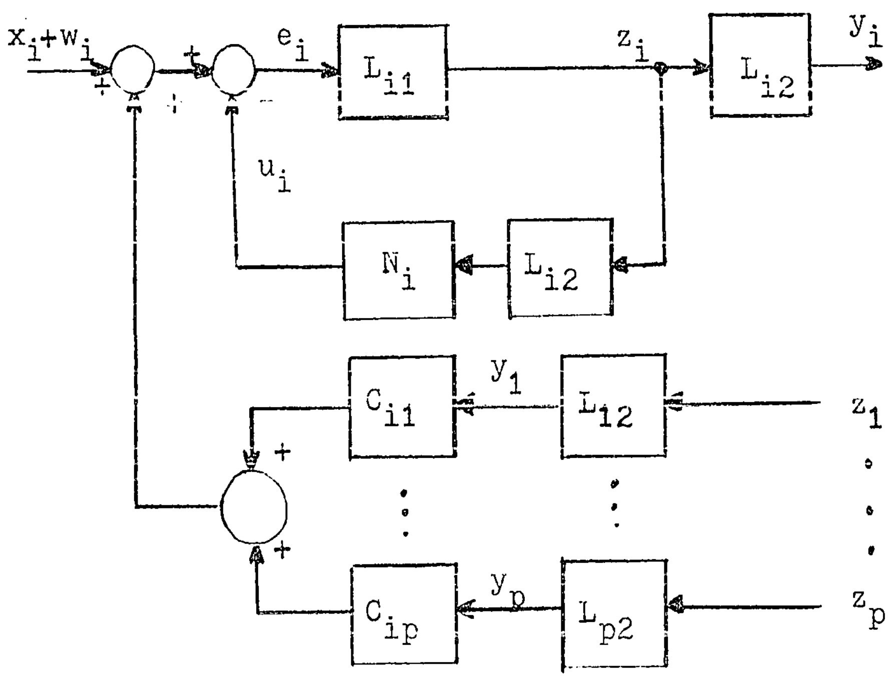

Figure 9b: Transformed system equivalent to interconnected system (14) of Figure 4b.

is of the Popov type, the operator $N_i L_{i2}$ is also inside the sector $\{0, b_i\}$ (see Lemma 2 of Reference 5). Therefore, using the basic definition of "inside a sector", we have

$$\langle v_{iT}, (N_i L_{i2} v_i)_T \rangle \geq b_i^{-1} \langle (N_i L_{i2} v_i)_T, (N_i L_{i2} v_i)_T \rangle$$
$$= b_i^{-1} \|(N_i L_{i2} v_i)_T\|^2 = b_i^{-1} \|y_{iT}\|^2 \tag{D2}$$

for all $T \in R$, $v_i \in L_{2e}$. Also, by assumption, the operator $L_{i1} - \delta_i + b_i^{-1}$ is positive, which means that

$$\langle (L_{i1} y_i)_T, y_{iT} \rangle \geq (\delta_i - b_i^{-1}) \langle y_{iT}, y_{iT} \rangle$$
$$= (\delta_i - b_i^{-1}) \|y_{iT}\|^2. \tag{D3}$$

Combining the last two results to obtain a lower bound on the last line of (D1) and using the triangle inequality and Schwarz's inequality to obtain an upper bound on the first line of (D1), we obtain

$$\|y_{iT}\| \left[\|(L_{i3} x_i)_T\| + \|(L_{i3} w_i)_T\| + \sum_{j=1}^{p} \|(L_{i3} c_{ij} y_j)_T\| \right]$$
$$\geq b_i^{-1} \|y_{iT}\|^2 + (\delta_i - b_i^{-1}) \|y_{iT}\|^2 = \delta_i \|y_{iT}\|^2$$

Using the definition of gain, this in turn implies that

$$\left\|(L_{i3}x_i)_T\right\| + \left\|(L_{i3}w_i)_T\right\| \geqslant \delta_i \left\|y_{iT}\right\| - \sum_{j=1}^{p} g(L_{i3}C_{ij}) \left\|y_{jT}\right\|$$

Introducing the column vectors $(LX)_T$, $(LW)_T$, and Y_T as in (1), this may be written as the matrix inequality $A\ Y_T \leqslant (LX)_T + (LW)_T$ where A is the test matrix of Theorem 10. If the successive principal minors of A are all positive, then A^{-1} exists and all its components are nonnegative. Boundedness of the relations which map inputs x_j into each output y_i then follows in the usual way, the only new point being that we restrict the inputs and reference signals so that $\dot{x}_i$ and $\dot{w}_i \in L_2$ (as well as x_i and w_i themselves) which implies that $L_{i3}x_i$ and $L_{i3}w_i \in L_2$. Boundedness of the relations which connect inputs x_j with errors e_i follows since the gains $g(L_i)$ and $g(C_{ij})$ must be finite under our assumptions (this is obvious upon inspection of Figure 4a). This completes the proof of Theorem 10.

Under the assumptions of Theorem 11, the interconnected system shown in Figure 4b and governed by the equations (14) is equivalent to the system shown in Figure 9b and governed by the following equations

$$e_i = x_i + w_i - u_i + \sum_{j=1}^{p} C_{ij} L_{j2} z_j, \tag{14'}$$

$$u_i = N_i y_i, \quad y_i = L_{i2} z_i, \quad z_i = L_{i1} e_i,$$

for $i = 1,2,\ldots,p$. For the moment, let

$$r_i = x_i + w_i + \sum_{j=1}^{p} C_{ij} L_{j2} z_j .$$

The first equation in (14') then reads $e_i = r_i - u_i$. Operating on both sides with L_{1i} and using (14') this becomes $z_i = L_{i1} r_i - L_{i1} u_i$. Truncating this equation at time $T \in R$ and taking the scalar product of the result with u_{iT}, we obtain

$$\langle (N_i L_{i2} z_i)_T, z_{iT} \rangle + \langle u_{iT}, (L_{i1} u_i)_T \rangle = \langle u_{iT}, (L_{i1} r_i)_T \rangle$$

Since we retain the assumptions of Theorem 10, equations (D2) and (D3) still hold. Using them to obtain a lower bound on the left-hand side of the preceding equation and using the Schwarz inequality to obtain an upper bound on the right-hand side, we have

$$\delta_i \, \|u_{iT}\|^2 = \delta_i \, \|(N_i L_{i2} z_i)_T\|^2$$

$$= b_i^{-1} \, \|(N_i L_{i2} z_i)_T\|^2 + (\delta_i - b_i^{-1}) \, \|(N_i L_{i2} z_i)_T\|^2$$

$$\leq \|u_{iT}\| \; \|(L_{i1} r_i)_T\|$$

which, in turn, implies

$$\delta_i \, \|u_{iT}\| \leq \|(L_{i1}r_i)_T\| \leq g(L_{i1}) \, \|r_{iT}\| \qquad \text{(D4)}$$

But, we also have $z_i = L_{i1}r_i - L_{i1}u_i$, which implies that

$$\|z_{iT}\| \leq g(L_{i1}) \left[\|r_{iT}\| + \|u_{iT}\| \right] \qquad \text{(D5)}$$

Combining (D4) and (D5), we obtain

$$\|z_{iT}\| \leq g(L_{i1}) \left[1 + g(L_{i1})/\delta_i \right] \|r_{iT}\| = D_i^{-1} \, \|r_{iT}\|$$

where the constant D_i is defined in the text (see p. 72). Recalling the definition of r_i and employing the triangle inequality and the definition of gain, this last result implies

$$D_i \, \|z_{iT}\| \leq \|x_{iT}\| + \|w_{iT}\| + \sum_{j=1}^{p} g(C_{ij}L_{j2}) \, \|z_{jT}\|.$$

Employing the, by now, familiar vector notation, this may be re-expressed as $A \, Z_T \leq X_T + W_T$ where A is the test matrix of Theorem 11. If the successive principal minors of A are all positive, then boundedness of the relations which connect inputs x_j to each z_i follows in the usual way. Boundedness of the relations which connect inputs x_j with each output y_i

and each error e_i then follows since our assumptions guarantee that the gains $g(L_{i2})$, $g(N_i)$, and $g(C_{ij}L_{j2})$ are all finite (this claim is obvious upon inspection of Figure 9b). This completes the proof of Theorem 11.

The system of Figure 4b and equations (14) is also studied in Theorem 12, but with slightly different assumptions. In order to discuss boundedness, this system too is transformed into the equivalent system shown in Figure 9b and described by equations (14'). From Lemma 2 of Reference 5, the resulting feedback operator N_iL_{i2} will be inside the sector $\{a_i, a_i+b_i\}$ if N_i is, since L_{i2} is a Popov multiplier. Introducing the transformed operators

$$-L'_{i1} = (-L_{i1}^{-1} - a_i I)^{-1}$$

$$(N_iL_{i2})' = N_iL_{i2} - a_i I$$

and performing the corresponding transformation (discussed in Appendix A) upon each subsystem, results in a system of <u>exactly</u> the same form as shown in Figure 9b with L_{i1} replaced by L'_{i1} and N_iL_{i2} replaced by $(N_iL_{i2})'$. Accordingly, the transformed operator L'_{i1} will be such that $L'_{i1} + b_i^{-1} - \delta_i$ is positive and $(N_iL_{i2})'$ will be inside the sector $\{0, b_i\}$. Thus, the system has been cast into exactly the same form as the transformed system discussed in connection with Theorem 11. The remainder of the proof of Theorem 12 is then a straightforward adaptation of that of Theorem 11, the appropriate test matrix being the one given in the statement of Theorem 12.